Vol.2 No. Mathematics 1(2025),pp.6-23, African Scientific Annual Review www.asarev.net

# On the Exponential and Approximate Point Spectrum of a Bounded Linear Operator in Banach Spaces

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***Original Research Article***
*DOI: https://doi.org/10.51867/Asarev.Maths.2.1.2*

## Abstract

This paper investigates the exponential and approximate point spectrum of a bounded linear operator in Banach spaces. We define the operator $exp(T)$ for a bounded linear operator $T$ in a Banach space $X$ and determine if $exp(T)$ is invertible. We show that if $S$ is a bounded linear operator in $X$ and $S$ commutes with $T$ then $exp(S+T)$ equals $exp(T).exp(S)$. If H is a Hilbert space and $T$ is a bounded linear operator in H which is normal such that $T$ commutes with a bounded linear operator $S$ in $H$, then $S$ commutes with the adjoint of $T$. The converse of this statement holds. For a bounded linear operator $A$ in $H$, the boundary of the spectrum of $A$ is contained in subset of the approximate point spectrum of $A$. Also, if $A$ and $B$ are bounded linear operators in $H$ which are similar, then $A$ and $B$ have the same spectrum, point spectrum, approximate point spectrum and compression spectrum. Using the Spectral Mapping Theorem, we have shown that the exponential and approximate point spectrum are independent of the Banach space.

**Keywords**: Exponential, approximate point spectrum, compression spectrum, Banach spaces

## 1 Introduction

The exponential operator and spectrum analysis are fundamental concepts in functional analysis and operator theory. By studying the behaviour of exponential operator, researchers gain deeper understanding of the structural properties of a bounded linear operators on Banach spaces. This forms basis of many theorotical developments in the field. Several substantial attempts has been made to transfer some of the important features of the spectral theory of normal operators from the realm of Hilbert spaces to the more general setting of Banach spaces. The most prominent and ambitious step in the early development of abstract spectral theory was the systematic investigation of spectral operator on Banach spaces that was initiated by Dunford [2]. The spectrum of a bounded linear operator on Banach Spaces from the view point of local spectral theory has been investigated in [11], whereby the local spectral properties may be used to identify certain parts of the spectrum. The exponential stability of operators has been studied in [14]. Mohammed and Ahmed in [9] studied some properties and the spectrum of exponential operators on a Hilbert space. They considered several properties such as linearity, norm, analyticity and even the semi group properties and the inverse. They concluded that the above properties can be passed on from the original operator to the exponential operator.However the later may not have an inverse defined for all bounded linear operators.

To extend the concept of exponential operators on Banach spaces, Gill et al. in [5] developed the concept of semigroups with an approach to operator theory. They exploited this relationship to transfer the theory of semigroups of operators developed for Hilbert spaces to Banach spaces. Their result was complete for uniformly convex Banach spaces. Van Neerven in [13] characterized exponential stability of semigroup of operators in terms of its action by convolution on vector valued function spaces over $R_+$, the study of the characterization of these operators provided the fact that the characterization is not limited to stability of operators.This provides the need to establish other properties to compute our results.

Gavrilyuk and Makarov [6] proposed a new exponentially convergent algorithms for the operator exponential generated by a strongly positive operator $A$ in a Banach space $X$. These algorithms were based on representations by a Dunford-Cauchy integral along paths enveloping the spectrum of A combined with a proper quadrature involving a short sum of resolvent. A parabola and a hyperbola are analyzed as the integration paths, and scales of dependence on smoothness of initial data, i.e., of the initial vector and of the in homogeneous right-hand side, are obtained. One of the algorithm possesses an exponential convergence rate for the operator exponential $e^{-At}$ for all $t \geq 0$ including the initial point. This allows one to construct an exponentially convergent algorithm for in homogeneous initial value problems.

Sourour [12] studied strongly continuous semigroups $T(t) : t \geq 0$ of scalar type operators on a Banach space and extended some well-known results about semigroups of normal or self adjoint operators on a Hilbert space. The main result was that if $T(t) : t \geq 0$ is a strongly continuous semigroup of scalar type operators on a weakly complete Banach space X, and if the resolutions of the identity for $T(t)$ are uniformly bounded in norm, then the infinitesimal generator is scalar type. Moreover, there exists a countably additive spectral measure $K(.)$ such that $T(t) = \int exp(\lambda t) dK(\lambda)$,for $t \geq 0$. Similar spectral representations were given for representations of locally compact abelian groups and for semi groups of unbounded operators. Connections with the theory of hermitian and normal operators on Banach spaces were established. It was further shown that $R$ is the infinitesimal generator of a semi groups of hermitian operators on a Banach space if and only if $iR$ is the generator of a group of isometrics.

## 2 Basic Definitions and Preliminary Results

In this section, we give some basic definitions and results that are useful in the sequel. Most of these definitions standard and can be found in [7, 4, 13, 15].

**Theorem 1.** *Projection Theorem*
Let $m$ be a closed linear subspace of a Hilbert space $H$ then

$$H = m \bigoplus m^{\perp} \textit{ where } m^{\perp} = \{y \in H :< y, x >= 0 \ \forall \ x \in H\}$$

**Theorem 2.** *Rieze Representation Theorem*
Let $H$ be a Hilbert space and $f \in H^*$ i.e $f$ is a bounded linear functional on $H$. Then there exist a unique element $y_f \in H$ such that;

$$f(x) =< x, y_f > \ \ \forall x \in H.$$

*Moreover,*

$$\|f\| = \|y_f\|$$

**Theorem 3.** *Let $H, K$ be Hilbert spaces over $\mathbb{C}$ and $T$ be a bounded linear operator i.e $T \in B(H, K)$. Define a map $f_y : H \to \mathbb{C}$ for any fixed $y \in K$ such that $f_y(x) =< Tx, y >$. Then,*

*i. $f_y$ is a linear functional on $H$.*

*ii. $f_y$ is bounded.*

**Theorem 4.** *Consider the map $T^* : y \to y^*$, i.e $K \to H$ such that $T^*(y) = y^*$. Then*

*i. $T^*$ is linear*

*ii. $T^*$ is bounded*

**Definition 2.1.**
If $T \in B(H, K)$, then there exist a unique $T^* \in B(K, H)$ such that $T^*$ is called the (Hilbert) adjoint of $T$.

**Definition 2.2.**
An operator $T \in B(H)$ is said to be invertible if there exists an operator $S \in B(H)$ such that

$$ST = TS = I_H$$

where $I_H$ is the identity map on $H$.

**Proposition 1.** Let $H$ be a Hilbert space and $S, T \in B(H)$. The following holds:

**i** $)^* = 0$ where 0 is the zero operator.

**ii** $I^* = I$,t where $I$ is the identity operator on $H$

**iii** $(\sigma T)^* = \bar{\sigma} T^* \;\forall\; \sigma \in \mathbb{C}$

**iv** $(S+T)^* = S^* + T^*$

**v** $(ST)^* = T^* S^*$

**vi** $T$ is invertible if and only if $T^*$ is invertible and then $(T^*)_{-1} = (T^{-1})$

**Lemma 1.**
Let $T \in B(H)$, then $\|T^*T\| = \|T\|^2$

**Definition 2.3.**
$T \in B(H)$ is called a contraction if $\|T\| \leq 1$

**Proposition 2.**
An operator $T$ in $H$ (with domain as the linear subspace $D_T$) is said to be bounded from below if there is a positive constant $B$ such that $\|Tx\| \geq B\|x\| \;\forall x \in D_T$

**Proposition 3.** Let $T \in B(H)$ and be bounded from below. Then range $T$ is closed.

**Proposition 4.**
$T \in B(H)$ is invertible if and only if $T$ is bounded from below and $\Re(T)$ is dense in $H$.

**Theorem 5.** *Banach inverse theorem*
Let $X$ be a Banach space and $T \in B(X)$ which is $1-1$ and onto. Then the set inverse $T^{-1} \in B(X)$ . i.e $T$ is invertible.

**Proposition 5.**
Let $X$ be a Banach space and $(x_n)$ be a sequence of elements of $X$. If $\sum_{n\in\mathbb{N}} x_n$ is absolutely convergent, then $\sum x_n$ converges to an element of $X$ in the norm of $X$. Moreover, if $\sum x_n$ converges to $x$, we have

$$\sum_{n\in\mathbb{N}} \|x_n\| \leq \|x\|$$

.

**Proposition 6.**
If $T \in B(H)$ and $\|I - T\| < 1$, then $T$ is invertible

**Definition 2.4.**
Let $T \in B(H)$ and $M$ be a closed linear subspace of $H$. We say that $M$ is invariant under $T$ if $T_x \in M \;\forall\; x \in M$

**Proposition 7.**
Let $M$ be invariant under $T$,($M$ is a closed linear subspace of $H$). Then $M^\perp$ is invariant under $T^*$, conversely $M^\perp$ is invariant under $T^* \Rightarrow M$ is invariant under $T$. Thus $M$ is invariant under $T$ implies that $M^\perp$ is invariant under $T^*$.

**Definition 2.5.**
Let $T \in B(H)$ and $M$ be a closed linear subspace of $H$. We say that $M$ reduces $T$ if $M$ and $M^{\perp}$ are both invariant under $T$.

**Proposition 8.**
Let $T \in B(H)$ and $M$ be a subspace of $H$. Let $P$ be the orthogonal projection on $H$ onto $M$. Then

**1** $M$ is invariant under $T$ if and only if $PTP = TP$

**2** *$M$ reduces $T$ if and only if $T \leftrightarrow P$ i.e, $TP = PT$*

# 3 Operator $e^T$ for a bounded $T$ in a Banach space $X$

In this section, we study the exponential and approximate point spectrum of an operator $e^T$ for a bounded $T$ in Banach spaces $X$.

## 3.1 Operator $e^T$

**Definition 3.1.** *Let $X$ be a Banach space and $T \in B(X)$. The expression $e^T$ stands for the infinite series*

$$I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \frac{T^4}{4!} + \dots$$

*where $I$ is the identity operator on $X$. It is clear that $e^T \in B(X)$.*
*Indeed, we first have*

$$\|T^n\| \leq \|T\|^n \quad \forall n \in \mathbb{N} \quad (\textit{for } \left\|T^2\right\| \leq \|T\| \, \|T\| = \|T\|^2 \textit{, etc., use induction on } n).$$

*Now, the expression*

$$I + \|T\| + \frac{\|T\|^2}{2!} + \frac{\|T\|^3}{3!} + \frac{\|T\|^4}{4!} + \dots + \frac{\|T\|^n}{n!} + \dots$$

*converges in $\mathbb{R}$ and its sum is $e^{\|T\|} \in \mathbb{R}$.*
*Thus,*

$$\sum_{n=0}^{\infty} \frac{T^n}{n!} \quad (T^0 = I)$$

*is absolutely convergent in $B(X)$.*
*For*

$$\left\| \sum_{n=0}^{\infty} \frac{T^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \left\| \frac{T^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{\|T^n\|}{n!} < \infty$$

*and hence $\sum_{n=0}^{\infty} \frac{T^n}{n!}$ converges in the norm of $B(X)$ to an element of $B(X)$ which we denote by $e^T$.*

**Lemma 2.** $T \in B(X) \Rightarrow \left\|e^T\right\| \leq e^{\|T\|}$.

**Lemma 3.** $e^0 = I$ *(for $0 \in B(X)$).*
Clearly, put $T = 0$ in $e^T = I + T + \frac{T^2}{2!}$ and we get $e^0 = I$.

**Proposition 9.** Let $T \in B(X)$

1. $e^T$ is invertible
2. $e^T e^{-T} = I$
3. If $S \in B(X)$ and $S \leftrightarrow T$ then $e^{S+T} = e^S e^T$

*Proof.* We prove (3) and subsequently (1) and (2) follow at once. By definition

$$e^T = I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \dots$$

$$e^S = I + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \dots$$

$$e^{S+T} = I + (S+T) + \frac{(S+T)^2}{2!} + \frac{(S+T)^3}{3!} + \dots$$

Let

$$a_n = I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \dots + \frac{T^n}{n!}$$

$$b_n = I + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \dots + \frac{S^n}{n!}$$

$$c_n = I + (S+T) + \frac{(S+T)^2}{2!} + \frac{(S+T)^3}{3!} + \dots + \frac{(S+T)^n}{n!}$$

and

$$\tilde{a}_n = I + \|T\| + \frac{\|T\|^2}{2!} + \dots + \frac{\|T\|^n}{n!}$$

$$\tilde{b}_n = I + \|S\| + \frac{\|S\|^2}{2!} + \dots + \frac{\|S\|^n}{n!}$$

$$\tilde{c}_n = I + (\|S\| + \|T\|) + \frac{(\|S\| + \|T\|)^2}{2!} + \dots + \frac{(\|S\| + \|T\|)^n}{n!}$$

for $n \in \mathbb{N}$.
By our earlier discussion, we note that

$$a_n \to e^T, \quad b_n \to e^S, \quad c_n \to e^{S+T} \text{ in } B(X)$$

$$\tilde{a}_n \to e^{\|T\|}, \quad \tilde{b}_n \to e^{\|S\|}, \quad \tilde{a}_n + \tilde{b}_n \to e^{\|S\|+\|T\|} \text{ in } \mathbb{R}$$

Now

$$a_n b_n - c_n = (a_n)(b_n) - (c_n)$$

In expanding $(S+T)^2, (S+T)^3$, etc., we use $S \leftrightarrow T$. So that we have

$$\begin{aligned}
(S+T)^2 &= (S+T)(S+T) \\
&= S^2 + ST + TS + T^2 \\
&= S^2 + 2ST + T^2 \\
(S+T)^3 &= S^3 + 3S^2T + 3ST^2 + T^3
\end{aligned}$$

Hence

$$a_n b_n - c_n = \sum_{i=1}^{n} a_i k T^i S^k$$

where $(a_i k)$ is a $n \times n$ matrix with positive entries on the main diagonal and below, and 0's elsewhere.

For simplicity, let $n = 3$ so,

$$\left(1+a+\frac{a^2}{2!}+\frac{a^3}{3!}\right)\left(1+b+\frac{b^2}{2!}+\frac{b^3}{3!}\right)-\left(1+(a+b)+\frac{(a+b)^2}{2!}+\frac{(a+b)^3}{3!}\right)=$$

$$\left(1+b+\frac{b^2}{2!}+\frac{b^3}{3!}\right)+\left(a+ab+\frac{ab^2}{2!}+\frac{ab^3}{3!}\right)+\left(\frac{a^2}{2!}+\frac{a^2b}{2!}+\frac{a^2b^2}{2!2!}+\frac{a^2b^3}{2!3!}\right)+$$

$$\left(\frac{a^2}{3!}+\frac{a^3b}{3!}+\frac{a^3b^2}{3!2!}+\frac{a^3b^3}{3!3!}\right)-\left(1+a+b+\frac{a^2}{2!}+\frac{b^2}{2!}+\frac{2ab}{2!}+\frac{a^3}{3!}+\frac{b^3}{3!}+\frac{3a^2b}{3!}+\frac{3ab^2}{3!}\right)=$$

$$ab^3\frac{3!}{3!}+a^2\left(b^2\frac{2!}{2!2!}+b^3\frac{3!}{2!3!}\right)+a^3\left(b^3\frac{3!}{3!}+b^2\frac{3!}{2!2!}+b^3\frac{3!}{3!3!}\right)=$$

$$\sum_{i=1}^{3} c_i k a_i b_k$$

$$c_i k a_i b_k = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2!} & 1 & \frac{1}{2!} \\ \frac{1}{3!} & \frac{1}{2!2!} & \frac{1}{3!} \end{pmatrix}$$

Therefore, $\|a_n b_n - c_n\| = \left\|\sum_{i=1}^{n} c_i k a_i b_k\right\| \le \sum_{i=1}^{n} \|c_i k\| \|a_i\| \|b_k\| \le \sum_{i=1}^{n} c_i k \|T\|^i \|S\|^k = \tilde{a}_n \tilde{b}_n - \tilde{c}_n$

As $n \to \infty$

$$\tilde{a}_n \tilde{b}_n - \tilde{c}_n \to e^{\|T\|} e^{\|S\|} - e^{\|T\|+\|S\|} = e^{\|T\|} e^{\|S\|} - e^{\|T\|} e^{\|S\|} = 0$$

So $\|a_n b_n - c_n\| \le \tilde{a}_n \tilde{b}_n - \tilde{c}_n \to 0$ as $n \to \infty$

But $a_n \to e^T$, $b_n \to e^S$, $c_n \to e^{T+S}$, so $\|a_n b_n - c_n\| \to \left\|e^S e^T - e^{T+S}\right\| = 0$ as $n \to \infty$

Therefore, $e^T e^S = e^{T+S}$ when $T \leftrightarrow S$.

Let $T \in B(X)$ so $-T \in B(X)$, $e^T, e^{-T} \in B(X)$ and $T \leftrightarrow -T$

Therefore, $e^T e^{-T} = e^T e^{-T} = e^{T-T} = e^0$, i.e., $e^T$ is invertible and its inverse is $e^{T^{-1}} = e^{-T}$.

Consider a Hilbert space $H$. Each $T \in B(H)$ has a unique adjoint $T^*$. For $T \in B(H)$, $e^T \in B(H)$. □

**Lemma 4.**

$$(e^T)^* = e^{T^*}$$

*Proof.*

$$e^T = I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \ldots + \frac{T^n}{n!} + \ldots$$

$$e^{T^*} = I + T^* + \frac{(T^*)^2}{2!} + \frac{(T^*)^3}{3!} + \ldots + \frac{(T^*)^n}{n!} + \ldots$$

Let

$$A_n = I + T + \frac{T^2}{2!} + \ldots + \frac{T^n}{n!}$$

$$A_n^* = I + T^* + \frac{(T^*)^2}{2!} + \ldots + \frac{(T^*)^n}{n!}$$

for all $n \in \mathbb{N}$. Then $A_n \to e^T$, $A_n^* \to e^{T^*}$ in $B(H)$. $(A_n, B_n \in B(H))$.

If $A_n$ is a sequence of elements of $B(H)$ which converges to $A \in B(H)$ then, $(A_n^*)$ converges to $A^*$ in $B(H)$. Indeed,

$$\|A_n - A\| \to 0 \text{ as } n \to \infty$$

We must show that

$$\|A_n^* - A^*\| \to 0 \text{ as } n \to \infty$$

Clearly $A_n - A \in B(H)$ so $(A_n - A)^*$ exist in $B(H)$, i.e. $A_n^* - A^*$ exist in $B(H)$.
Moreover, $\|A_n - A\| = \|(A_n - A)^*\| = \|A_n^* - A^*\|$. Therefore, $\lim_{n\to\infty} \|A_n - A\| = 0 \Rightarrow \lim_{n\to\infty} \|A_\infty^n - A_\infty\| = 0$ i.e. $A_\infty^n \to A_\infty$ in $B(H)$. By the result proved it follows that since $A_n \to e^T$. So

$$(A_n)_\infty \to e^{T^*}.$$

But

$$A_n^* \to e^{T^*}$$

By the uniqueness of the limit, it follows that $(e^T)^* = e^{T^*}$ □

**Proposition 10.** Let $H$ be a complex Hilbert space and $T \in B(H)$. If $\langle Tx, x\rangle = 0$ for all $x \in H$, then $T = 0$.

*Proof.* First we know that if we have $\langle Tx, y\rangle = 0$ for all $x, y \in H$, the conclusion $T = 0$ is immediate, for $Tx \perp y$ for all $x \in H$ and $y \in H$.

$$Tx \perp H \implies Tx = \overline{0} \quad \forall x \in H \implies T = 0, \text{ the zero operator}$$

Let

$$\Phi(x, y) = \langle Tx, y\rangle : \Phi \to \text{sesquilinear form}$$

$$\widehat{x} \to \text{associated quadratic form}$$

$$\Phi(\widehat{x}) = \langle Tx, x\rangle$$

Now by the polarization identity,

$$\Phi(x, y) = \frac{1}{4}\left[\Phi(\widehat{x} + y) - \Phi(\widehat{x} - y) + i\Phi(\widehat{x} + iy) - i\Phi(\widehat{x} - iy)\right]$$

i.e.,

$$\Phi(x, y) = \frac{1}{4}[\Phi(\widehat{x} + y) - \Phi(\widehat{x} - y) + i\Phi(\widehat{x} + iy) - i\Phi(\widehat{x} - iy)]$$

Let $Z = (x + y, x - y, x + iy, x - iy)$.

$$\langle TZ, Z\rangle = 0 \quad \forall Z \in H$$

$$\implies \langle Tx, y\rangle = 0 \quad \forall x, y \in H$$

$$\implies T = 0.$$

The above result is not true in real Hilbert spaces. □

**Example 3.1.** Let $H = \mathbb{R}^2$ with the usual inner product

$$\langle (x, y), (z, w) \rangle = xz + yw.$$

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y) = (-y, x).$$

Therefore,

$$\langle T(x, y), (x, y) \rangle = \langle (-y, x), (x, y) \rangle = -yx + xy = 0 \quad \forall (x, y) \in H.$$

But $T \neq 0$.

**Definition 3.2.** An operator $T \in B(H)$ is called *Isometric* if $\|Tx\| = \|x\|$ for all $x \in H$.

**Lemma 5.** For a $T \in B(H)$ the following conditions are equivalent:

(i) $T$ is isometric.

(ii) $T^*T = I$

*Proof.* (i) $\Rightarrow$ (ii):

By (i),

$$\|Tx\|^2 = \|x\|^2 \quad \forall x \in H$$

which implies

$$\langle Tx, Tx \rangle = \langle x, x \rangle \quad \forall x \in H$$

or equivalently,

$$\langle T^*Tx, x \rangle = \langle Ix, x \rangle \quad \forall x \in H$$

Thus, $T^*T = I$ by a previous result (where $H$ is assumed to be a complex Hilbert space). Indeed,

$$\langle T^*Tx, x \rangle - \langle Ix, x \rangle = 0 \quad \forall x \in H$$

which implies

$$\langle T^*Tx - Ix, x \rangle = 0 \quad \forall x \in H$$

and further implies

$$\langle (T^*T - I)x, x \rangle = 0 \quad \forall x \in H$$

Therefore, $S = 0$. □

**Lemma 6.** If $H$ is a complex Hilbert space, $S, T \in B(H)$, and $\langle Sx, x \rangle = \langle Tx, x \rangle$ for all $x \in H$, then $S = T$.

*Proof.* For all $x$,

$$\langle Sx - Tx, x \rangle = 0 \quad \text{i.e.,} \quad \langle (S - T)x, x \rangle = 0$$

Therefore, $S - T = 0$, i.e., $S = T$.

(ii) $\Rightarrow$ (i):

Now for all $x$,

$$\begin{aligned} \|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle = \langle Ix, x \rangle \quad (\text{since } T^*T = I) \\ = \|x\|^2 \end{aligned}$$

i.e., $\|Tx\| = \|x\|$, i.e., $T$ is isometric. □

**Definition 3.3.** An operator $T \in B(H)$ is called *Unitary* if $TT^* = T^*T = I$. Thus, $T \leftrightarrow T^*$ and $T^*T = I$. Clearly, $T$ is invertible and $T^{-1} = T^*$.

**Definition 3.4.** $T \in B(H)$ is called a *Normal operator* if $T \leftrightarrow T^*$.

**Lemma 7.** For $T \in B(H)$, the following conditions are equivalent:

(i) $T$ is normal.

(ii) $\|Tx\| = \|T^*x\|$ for all $x \in H$.

*Proof.* $(i) \Rightarrow (ii)$

Now $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle$ (for $T$ is normal by (i)) $= \langle T^*x, T^*x\rangle = \|T^*x\|^2$ for all $x \in H$. Therefore, $\|Tx\| = \|T^*x\|$ for all $x \in H$, which is (ii).

$(ii) \Rightarrow (i)$

$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 = \|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle (T^*)^*T^*x, x\rangle = \langle TT^*x, x\rangle$ for all $x \in H$. Therefore, by a previous result, $T^*T = TT^*$, i.e., $T \leftrightarrow T^* \Rightarrow T$ is normal. □

## 3.2 Normal operators in $B(H)$

A self-adjoint operator $T$ is normal for $T = T^*$, so since $T \leftrightarrow T^*$, $T \leftrightarrow T^* = T$. A unitary operator $U \in B(H)$ is normal for $UU^* = I = U^*U \Rightarrow U \leftrightarrow U^*$.

**A special observation:** Let $T \in B(H)$ be normal, we can write,

$$T = \frac{1}{2}(T + T^*) \quad \text{(A)}$$

$$+\frac{i}{2}(T - T^*) \quad \text{(B)}$$

for all $T \in B(H)$. So $T = A + iB$. Now $A, B \in B(H)$ and $A^* = \left(\frac{1}{2}(T + T^*)\right)^* = \frac{1}{2}(T^* + T^{**}) = \frac{1}{2}(T + T^*) = A$. $B^* = \left(\frac{i}{2}(T - T^*)\right)^* = \frac{-i}{2}(T^* - T^{**}) = \frac{-i}{2}(T - T^*) = B$.

Thus, both $A$ and $B$ are self-adjoint. But $T$ is normal, so $T \leftrightarrow T^*$, i.e., $T^*T = T^*T$. Since $T = A + iB$, so $T^* = (A + iB)^* = A^* - iB^* = A - iB$.

Therefore, $TT^* = (A+iB)(A-iB) = A^2 - iAB + iBA + B^2$ and $T^*T = (A-iB)(A+iB) = A^2 + iAB - iBA + B^2$. Thus, $TT^* = T^*T \Rightarrow A^2 - iAB + iBA + B^2 = A^2 + iAB - iBA + B^2 \Rightarrow 2iAB - 2iBA = 0 \Rightarrow 2iAB = 2iBA \Rightarrow AB = BA$, i.e., $A \leftrightarrow B$.

Thus, every normal $T \in B(H)$ can be uniquely expressed as $A + iB$, where both $A$ and $B$ are self-adjoint and $A \leftrightarrow B$. The uniqueness of this decomposition follows since if $T = A + iB$ is such a decomposition, then

$$T^* = (A + iB)^* = A^* - iB^* = A - iB$$

solving the operator equations

$$T = A + iB$$
$$T^* = A - iB$$

we get unique solutions

$$A = \frac{T + T^*}{2}$$
$$B = \frac{1}{2i}(T - T^*)$$

### 3.3 Fuglede's Theorem

**Theorem 6.** If $T \in B(H)$ is normal and $T$ commutes with $S \in B(H)$, then $T^*S = ST^*$, i.e., $S \leftrightarrow T^*$.

*Proof.* Since $T \leftrightarrow S$, $T^n \leftrightarrow S$ for all $n \in \mathbb{N}$. Indeed, the result is obviously valid when $n = 1$. Suppose (Induction hypothesis) $T^{n-1} \leftrightarrow S$, then

$$T^nS = (TT^{n-1})S = T(T^{n-1}S) = T(ST^{n-1}) = (TS)T^{n-1} = (ST)T^{n-1} = ST^n$$

Therefore, $T^n \leftrightarrow S$ for all $n = 0, 1, 2, ...$

For any $z \in \mathbb{C}$, consider the operator $e^{i\bar{z}T} \in B(H)$. Note $e^{i\bar{z}T} = \sum_{n=0}^{\infty} \frac{(i\bar{z}T)^n}{n!}$. Since $S \leftrightarrow T^n$ for all $n = 1, 2, 3, \ldots$, we obtain

$$Se^{i\bar{z}T} = e^{i\bar{z}T}S \quad \ldots (1)$$

{For if $A_n = \sum_{m=0}^{n} \frac{(imz\overline{T})^m T^m}{m!}$, then $S \leftrightarrow A_n$ for each $n \in \mathbb{N}$. Now, $A_n \to e^{i\bar{z}T}$ as $n \to \infty$. As $S \leftrightarrow A_n$ for each $n \in \mathbb{N}$, $S \leftrightarrow \lim_{n\to\infty} A_n = e^{i\bar{z}T}$ } $\Rightarrow S = e^{-i\bar{z}T}Se^{i\bar{z}T}$ (from (1)).

Now consider the operator

$$e^{-izT^*}Se^{izT^*} = e^{-izT^*}(e^{i\bar{z}T}Se^{i\bar{z}T})e^{izT^*} \quad \text{(from (1))}$$
$$= e^{-i(zT^*+\bar{z}T)}Se^{i(zT+\bar{z}T^*)}$$

Since $T \leftrightarrow T^*$,

$$e^{i(\bar{z}T+zT^*)}(e^{i(\bar{z}T+zT^*)})^* = e^{-i(zT^*+\bar{z}T)}e^{i(\bar{z}T+zT^*)} = e^0 = I$$
$$= e^{i(\bar{z}T+zT^*)}(e^{-i(\bar{z}T+zT^*)})^*$$
$$\Rightarrow \text{the operator } e^{i(\bar{z}T+zT^*)} \text{ is unitary and its norm is } 1$$

Thus,

$$e^{(-i(zT^*+\bar{z}T)}Se^{i(zT^*+\bar{z}T)}$$

is $U^*SU$ where $U$ is the unitary operator $e^{i(\bar{z}T+zT^*)}$. It follows that

$$\|e^{-izT^*}Se^{izT^*}\| = \|USU^*\| \leq \|U\|\|S\|\|U^*\| = \|S\|$$

Hence, for any $x, y \in H$,

$$|\langle e^{-izT^*}Se^{izT^*}x, y\rangle| \leq \|e^{-izT^*}Se^{izT^*}\|\|x\|\|y\| \leq \|S\|\|x\|\|y\|$$

Now, $\langle e^{-izT^*}Se^{izT^*}x, y\rangle$ is analytic in $z$ and bounded in the entire complex plane, i.e., it is a bounded function. Hence, by Liouville's theorem, the function must be constant on $\mathbb{C}$. Setting, in particular, $z = 0$, we get

$$e^{-izT^*}Se^{izT^*} = ST^*$$

i.e., $Se^{izT^*} = e^{izT}S$. Equating coefficients of like powers of $z$ on both sides, we get

$$ST^* = T^*S$$

$$S(T^*)^2 = (T^*)^2S$$

etc. ∎

**Proposition 11.** *Let $S \in B(H)$ and $T \in B(H)$. If $T \leftrightarrow S$, it does not follow that $T^* \leftrightarrow S$. However, it is true that $T^* \leftrightarrow S^*$ for $TS = ST \Rightarrow (TS)^* = (ST)^*$ i.e., $S^*T^* = T^*S^*$ i.e., $S^* \leftrightarrow T^*$. However, if $T \leftrightarrow S \rightarrow T^* \leftrightarrow S^*$, then since $T \rightarrow T$, so $T^* \leftrightarrow T$ i.e., $T$ is normal. Fuglede's theorem is a converse of this, i.e., $T$ is normal and $T \leftrightarrow S \rightarrow T^* \leftrightarrow S^*$.*

# 4 Approximate Point Spectrum and Similar Operators

**Definition 4.1.** A number $\lambda \in \mathbb{C}$ is said to belong to the resolvent set $\rho(T)$ of an operator $T$ if $(T - \lambda I)$ is invertible. The complement of $\rho(T)$ in $\mathbb{C}$ is called the spectrum of $T$ and we represent it by $\sigma(T)$. Therefore, $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)$ is not invertible (in the operator sense)$\}$. We had an earlier result: $T \in B(H)$ is invertible if and only if $T$ is bounded from below and $\Re(T)$ is dense in $H$.

So $(T - \lambda I)$ is invertible if and only if $T - \lambda I$ is bounded from below and $\Re(T - \lambda I)$ is dense in $H$. The contrapositive statement is: $T - \lambda I$ is not invertible if and only if $T - \lambda I$ is not bounded from below or $\overline{\Re}(T - \lambda I) \neq H$. Therefore, $\lambda \in \sigma(T)$ then $T - \lambda I$ is not bounded from below or $\overline{\Re}(T - \lambda I) \neq H$.

Let $\pi\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not bounded from below$\}$ and $\Gamma\sigma(T) = \{\lambda \in \mathbb{C} : \Re(T - \lambda I)$ is not dense in $H\}$. $\pi\sigma(T)$ is called the approximate point spectrum of $T$ where $\Gamma\sigma(T)$ is called the compression spectrum of $T$. So $\sigma(T) = \pi\sigma(T) \cup \Gamma\sigma(T)$.

**Note**: The sets $\pi\sigma(T)$ and $\Gamma\sigma(T)$ may overlap.

It is clear that if we define $\lambda \in \mathbb{C}$ to be an approximate eigenvalue of $T$ if there exist a sequence $(x_n)$ of elements of $H$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|(T - \lambda I)x_n\| \rightarrow 0$ as $n \rightarrow \infty$, then $\pi\sigma(T)$ is the set of all approximate eigenvalues of $T$ for $(T - \lambda I)$ is bounded from below implying that there exist a real number $\epsilon > 0$ such that $\|(T - \lambda I)x\| \geq \epsilon\|x\|$ for all $x \in H$ such that $x \neq 0$.

Now $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Thus $\|(T - \lambda I)y\| \geq \epsilon$ for all $y \in H$ satisfying $\|y\| = 1$. Hence there exists a sequence $(x_n)$ such that $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \rightarrow 0$ as $n \rightarrow \infty$, i.e., $\lambda$ is not an approximate eigenvalue.

The converse can be similarly seen. If $\lambda$ is an eigenvalue of $T$, it is clear that $\lambda \in \Pi\sigma(T)$. Let $P\sigma(T) = \{\lambda : \lambda$ is an eigenvalue of $T\}$. Thus $P\sigma(T) \subset \pi\sigma(T)$. $P\sigma(T)$ is called the point spectrum of $T$. $\lambda \in P\sigma(T)$ implies there exists $x \in H$ such that $x \neq 0$ and $(T - \lambda I)x = 0$, i.e., $\|(T - \lambda I)x\| = 0$. If we take $x_n = \frac{x}{\|x\|}$ for all $n \in \mathbb{N}$, we note that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|(T - \lambda I)x_n\| = 0$, i.e., $x_n \rightarrow 0$ as $n \rightarrow \infty$, i.e., $\lambda \in P\sigma(T)$.

These are five disjoint sets, some of which may be void or not void, depending on $T$. The usual tradition in spectral theory is to have the following disjoint division of $\sigma(T)$:

- $p_{\sigma(T)}$ - Point Spectrum
- $\Gamma_{\sigma(T)} - p_{\sigma(T)}$ - Residual Spectrum of $T$ (denoted $R_{\sigma(T)}$).
- $\Pi_{\sigma(T)} - (p_{\sigma(T)} \cup \Gamma_{\alpha(T)})$ - Continuous Spectrum of $T$ (denoted $C_{\sigma(T)}$).

## We have the following result

Consider $\pi\sigma(T)$

$$\begin{aligned}\lambda \in P\sigma(T) &\Rightarrow T-\lambda I \text{ is not } 1-1\\ &\Rightarrow \sqcap(T-\lambda I) \neq \{\bar{0}\}\\ &\Rightarrow \sqcap(\perp)(T-\lambda I) \neq H\\ &\Rightarrow \Re_{T\infty-\lambda I} \neq H \left(\therefore \sqcap_A^{\perp} = \bar{\Re}A^* : A = T-\lambda I A^* = T^* - \bar{\lambda} I\right)\\ &: A = T-\lambda I, \quad A^* = T^* - \lambda I^*\\ &\Rightarrow \bar{\lambda} \in \Gamma\sigma(T^*)\\ &\Rightarrow \lambda \in \overline{\Gamma\sigma(T^*)}\end{aligned}$$

Where the bar denotes complex conjugation i.e $\bar{N} = \{Z \in \bar{\mathbb{N}}\}$.

The implications are reversible, and so we get

$$P\sigma(T) = \overline{\Gamma\sigma(T^*)}$$

Likewise, replacing $T$ by $T^*$, we get

$$P\sigma(T^*) = \overline{\Gamma\sigma(T^*)} = \Gamma\sigma(T)$$

Thus,

$$P\sigma(T) = \overline{\Gamma\sigma(T^*)}$$

and

$$P\sigma(T^*) = \overline{\Gamma\sigma(T)}$$

Consider $\sigma(T^*)$. We know a result:

$T^*$ is invertible if and only if $T$ is invertible. Now $T^*$ is invertible $\Rightarrow$ $T^*$ is bounded from below and $\bar{\Re}_{T^*} = H$. Therefore, $T^*$ is invertible $\Rightarrow$ $T^*$ is bounded from below.

$T$ is invertible $\Rightarrow$ $T$ is bounded from below.

Thus $T^*$ is invertible $\Rightarrow$ both $T$ and $T^*$ are bounded from below.

Conversely:

$T$ and $T^*$ are bounded from below $\Rightarrow$ $T^*$ is invertible. For since $T$ is bounded from below so $T$ is 1-1, i.e., $\sqcap_T = \{\bar{0}\}$. Therefore, $\sqcap_T^{\perp} = H$. But $\sqcap_T^{\perp} = \bar{\Re}_{T^*}$. Therefore, $\bar{\Re}_{T^*} = H$, thus $T^*$ is bounded from below and $\bar{\Re}_{T^*} = H$, i.e., $T^*$ is invertible. Hence we may assert (taking $T^* - \lambda I$ in place): $T^* - \lambda I$ is invertible $\Leftrightarrow$ both $T^* - \lambda I$ and $T - \lambda I$ are bounded below.

Thus $T^* - \lambda I$ is not invertible $\Leftrightarrow$ one of $T^* - \lambda I$ or $T - \bar{\lambda} I$ is not bounded from below.

Thus $\lambda \in \sigma(T^*) \Leftrightarrow \lambda \in \pi\sigma(T^*)$ or $\lambda \in \overline{\pi\sigma(T)}$, i.e., $\sigma(T^*) = \pi\sigma(T^*) \bigcup \overline{\pi\sigma(T)}$.

**Corollary 7.** $\sigma(T) = \pi\sigma(T) \bigcup \overline{\sigma(T^*)}$

**Proposition 12.** $\pi\sigma(T)$ is closed for $T \in B(\mathcal{H})$.

*Proof.* We shall show that $(\pi\sigma(T))^c$ is open.

Let $\lambda_0 \in (\pi\sigma(T))^c$, i.e., $\lambda_0 \notin \pi\sigma(T)$ hence $T - \lambda_0 I$ is bounded from below, i.e., $\exists\ \delta > 0$ such that $\|(T - \lambda_0)x\| \geq \delta\|x\| \quad \forall x \in \mathcal{H}$.

If $\lambda \in \mathbb{C}$, then

$$\begin{aligned}\|(T - \lambda_0 I)x\| &= \|(T - \lambda I)x + (\lambda - \lambda_0)x\| \\ &\leq \|(T - \lambda I)\| + |\lambda - \lambda_0|\|x\| \\ &\Rightarrow \|(T - \lambda I)x\| \geq \|(T - \lambda_0 I)x\| - |\lambda - \lambda_0|\|x\| \\ &\geq (\delta - |\lambda - \lambda_0|)\|x\| \quad \forall x \in \mathcal{H}.\end{aligned}$$

We can choose a $\lambda$ sufficiently close to $\lambda_0$ so that $\delta - |\lambda - \lambda_0|$ is positive, i.e., $T - \lambda I$ is bounded from below for $\lambda$ in a sufficiently small neighbourhood of $\lambda_0$. Thus for all $\lambda$ in a small neighbourhood of $\lambda_0$, $T - \lambda I$ is bounded from below.

$\therefore (\pi\sigma(T))^c$ is open since for a $\lambda_0 \in (\pi\sigma(T))^c$ there is a neighbourhood of $\lambda_0$ such that for all $\lambda$ in this neighbourhood, $T - \lambda I$ is bounded from below. Thus $\pi\sigma(T)$ must be closed. □

**Proposition 13.** If $A_n \in B(H)$ for all $n \in \mathbb{N}$ and are invertible, and $A \in B(H)$ is not invertible with $|A_n - A| \to 0$ as $n \to \infty$, then $0 \in \pi\sigma(A)$.

*Proof.* Since $A$, i.e., $A - 0I$ is not invertible, $0 \in \sigma(A) = \pi\sigma(A)$.

Hence, $0 \in \pi\sigma(A)$ or $0 \in \Gamma\sigma(A)$, so $\Re_A \neq H$, and hence there is a nonzero $x \in H$ such that $x \perp \Re_A$.

Define

$$x_n = A_n^{-1}x \left\|A_n^{-1}x\right\| \quad \text{for all } n \in \mathbb{N}$$

($x \neq 0$, $A_n \neq 0 \Rightarrow A_n^{-1}x \neq 0$ for $A_n$ is invertible).

Thus $|x| = 1$ for all $n \in \mathbb{N}$.

Now

$$\|A_n x_n - A x_n\| \leq \|A_n - A\|\|x_n\| = |A_n - A| \to 0 \quad \text{as } n \to \infty$$

Now

$$A_n x_n = A_n \left(A_n^{-1}x \frac{1}{\|A_n^{-1}x\|}\right) = x \frac{1}{\|A_n^{-1}x\|} \in \Re_A^{\perp} \quad (\text{since } x \in \mathrm{Re}\perp A)$$

But $Ax_n \in \Re_A$ so $A_n x_n \perp A x_n$.

Hence, by the Pythagorean theorem,

$$\|A_n x_n - A x_n\|^2 = \|A_n x_n\|^2 + \|A x_n\|^2 \geq \|A x_n\|^2$$

Since $\|A_n x_n - A x_n\| \to 0$ as $n \to \infty$, therefore $\|A x_n\| \to 0$ as $n \to \infty$, i.e., $\|(A - 0I)x_n\| \to 0$ and $\|x_n\| = 1$ for all $n \in \mathbb{N}$, hence $0 \in \pi\sigma(A)$. □

**Proposition 14.** Let $A \in B(H)$. Then $\delta\sigma(A) \subseteq \pi\sigma(A)$, where $\delta\sigma(A)$ is the boundary of the spectrum of $A$.

*Proof.* $\sigma(A)$ is closed. So $\delta\sigma(A) \subseteq \sigma(A)$.

Let $\lambda \in \delta\sigma(A)$. Hence, we can choose a sequence $(\lambda_n)$ of elements of $\rho(A)$ (complement of $\sigma(A)$) such that

$$\lambda_n \to \lambda \quad \text{as} \quad n \to \infty \quad \text{i.e.,} \quad |\lambda_n - \lambda| \to 0 \quad \text{as} \quad n \to \infty.$$

So $A - \lambda_n I$ are invertible for all $n \in \mathbb{N}$ and $A - \lambda I$ is not invertible ($A - \lambda_n I = A_n$ and $A - \lambda I = A$).

Now

$$\|(A - \lambda_n I) - (A - \lambda I)\| = |\lambda_n - \lambda|\|I\| = |\lambda_n - \lambda| \to 0 \quad \text{as} \quad n \to \infty$$

and each $A - \lambda_n I$ is invertible, $A - \lambda I$ is non-invertible. Hence, by the previous result, $0 \in \pi\sigma(A - \lambda I)$ i.e., there exists a sequence $(y_n)$ in $H$ with $\|y_n\| = 1$ and such that

$$\|(A - \lambda I) - 0I\|y_n\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{i.e.,} \quad \|(A - \lambda I)y_n\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{i.e.,} \quad \lambda \in \pi\sigma(A).$$

Hence, $\delta\sigma(A) \subseteq \pi\sigma(A)$. □

**Definition 4.2.** Let $A, B \in B(H)$. We say that $A$ is similar to $B$ if there exists an invertible $S \in B(H)$ such that $B = S^{-1}AS$. $A$ is similar to $B$ if and only if $B$ is similar to $A$.

$$\begin{aligned}
&B = S^{-1}AS \\
&\quad \Rightarrow SBS^{-1} = SS^{-1}ASS^{-1} \\
&\quad \Rightarrow SBS^{-1} = A \\
&\quad \Rightarrow (S^{-1})^{-1}BS^{-1} = A \quad \textit{and } S^{-1} \textit{ is also invertible} \\
&\quad \Rightarrow B \textit{ is similar to } A
\end{aligned}$$

**Example 4.1.** If $A$ and $B$ are similar, then $\sigma(A) = \sigma(B)$.

*Proof.* We will show that the resolvent sets are the same since $A$ is similar to $B$, i.e., $B = S^{-1}AS$ for an invertible $S \in B(H)$.

$$\begin{aligned}
B - \lambda I &= S^{-1}AS - \lambda I \\
&= S^{-1}(A - \lambda I)S
\end{aligned}$$

Hence $B - \lambda I$ is invertible if and only if $A - \lambda I$ is invertible, i.e., $\lambda \in \rho(B) = \rho(S^{-1}AS)$ if and only if $\lambda \in \rho(A)$. Thus, $\rho(S^{-1}AS) = \rho(A)$. Taking complements in $\mathbb{C}$, we get

$$\sigma(S^{-1}AS) = \sigma(A)$$

i.e., $\sigma(B) = \sigma(A)$. □

**Proposition 15.**
If B is similar to A then $P\sigma(B) = P\sigma(A)$

*Proof.*
$\lambda \in P\sigma(A) \Rightarrow \exists \;\; x \in H$ such that $x \neq \bar{0}$ and $A_x - \lambda_x = \bar{0}$. Hence with $B = S^{-1}AS$, where $S \in B(H)$is invertible , we have $S^{-1}(A_x - \lambda_x) = \bar{0}$.
i.e $S^{-1}A(S(S^{-1})x - \lambda(S^{-1}x) = \bar{0}$ (∗)
Since $x \neq \bar{0}$, $S^{-1}x \neq 0$ (since $S^{-1}x$ is invertible). Hence (*) implies $\lambda \in P\sigma(S^{-1}AS)$
Thus, $P\sigma(A) \subseteq P\sigma(S^{-1}AS) = P\sigma(B)$
Replacing A by B i.e $S^{-1}AS$ we get $P\sigma(S^{-1}AS) \subseteq P\sigma(S(S^{-1}AS)S^{-1}) = P\sigma(A)$. i.e $P\sigma(B) \subseteq P\sigma(A)$
Thus, $P\sigma(B) = P\sigma(A)$ □

**Proposition 16.** If $B$ is similar to $A$, Then $\pi\sigma(B) = \pi\sigma(A)$

*Proof.* Let $\lambda \in \sigma(A)$. Then there exists a sequence $(x_n) \in H$ such that $\|x_n\| = 1 \; \forall n \in \mathbb{N}$ and $\|(A - \lambda)x_n\| \to 0$ as $n \to \infty$, i.e., $(A - \lambda) \to 0$ as $n \to \infty$.

Let $B = S^{-1}AS$ for $S$ invertible. Then $S^{-1}(A - \lambda I)x_n \to 0$ as $n \to \infty$, i.e., $S^{-1}Ax_n - \lambda S^{-1} \to 0$ as $n \to \infty$, i.e., $S^{-1}AS(S^{-1}x_n - \lambda(S^{-1}x_n) \to 0$ as $n \to \infty$.

Now $x_n \neq 0$ so $S^{-1}x_n \neq 0$ i.e., $\|S^{-1}x_n\| \neq \bar{0}$ and $\|S^{-1}x_n\|$ is bounded away from 0. i.e., $\exists \sigma > 0$ such that $\|S^{-1}x_n\| \geq \sigma$. Let $S^{-1}x_n = y_n \; \forall n \in \mathbb{N}$. Therefore, $Syn = x_n$, $\|x_n\| = \|Syn\| \leq \|S\|\|y_n\|$ i.e., $\|y_n\| \geq 1$

$$\frac{\|S\|}{\|S\|}\|Syn\|$$

i.e., $\|S^{-1}x_n\| \geq \frac{1}{\|S\|}\|x_n\|$. So $\|S^{-1}x_n\|$ are bounded from below by $\frac{1}{\|S\|}$.

Hence $\left\| \frac{S^{-1}x_n}{\|S^{-1}x_n\|} \right\| = 1 \; \forall n \in \mathbb{N}$. Dividing (*) throughout by $\|S^{-1}x_n\|$ we have

$$S^{-1}AS\left(\frac{S^{-1}x_n}{\|S^{-1}x_n\|}\right) - \lambda\left(\frac{S^{-1}x_n}{\|S^{-1}x_n\|}\right) \to 0 \text{ as } n \to \infty$$

i.e., $\lambda \in \pi\sigma(S^{-1}AS)$. Thus $\pi\sigma(A) \subseteq \pi\sigma(S^{-1}AS) = \pi\sigma(B)$.

Replacing $A$ by $S^{-1}AS$ i.e., $B$ we have $\pi\sigma(S^{-1}AS) \subseteq \pi\sigma((S^{-1})^{-1}(S^{-1}AS)S^{-1})$ using the first part $= \pi\sigma(A)$. i.e., $\pi\sigma(S^{-1}AS) \subseteq \pi\sigma(A) \Rightarrow \pi\sigma(B) \subseteq \pi\sigma(A)$. Thus $\pi\sigma(A) = \pi\sigma(B)$. □

**Proposition 17.** If $B$ is similar to $A$, then $\Gamma\sigma(A) = \Gamma\sigma(B)$

*Proof.* Let $B = S^{-1}AS$ for an invertible $S \in B(H)$. Then $\forall \lambda \in \mathbb{C}$,

$$B - \lambda I = S^{-1}AS - \lambda I = S^{-1}(A - \lambda I)S$$

If $\lambda \in \sigma(A)$ then,

$$\mathrm{Re}\,(A - \lambda I) \neq H$$

and hence there exists an $x \in H$ such that $x \neq \overline{0}$ and $x \perp \text{Re}\,(A-\lambda I)$, $x \neq \text{Re}\,(A-\lambda I) \Rightarrow S^{-1}x \neq S^{-1}(\text{Re}\,(A-\lambda I))$ and $S^{-1} \neq \overline{0}$ (since $x \neq \overline{0}$ and $S^{-1}$ is invertible).

So $S^{-1}x \neq S^{-1}(A-\lambda I)S(H)$. Therefore, $\text{Re}\,S^{-1}(A-\lambda I)S$ is not dense in $H$. i.e., $S^{-1}(A-\lambda I)$ has range not dense in $H$. Therefore, $\lambda \in \Gamma\sigma(S^{-1}AS) = \Gamma\sigma(B)$. Thus $\Gamma\sigma(A) \subseteq \Gamma\sigma(B) = \Gamma\sigma(S^{-1}AS)$.

Likewise,

$$\Gamma\sigma(S^{-1}AS) \subseteq \Gamma\sigma((S^{-1})^{-1}(S^{-1}AS)S^{-1}) = \Gamma\sigma(A)$$

i.e.,

$$\Gamma\sigma(B) = \Gamma\sigma(S^{-1}AS) \subseteq \Gamma\sigma(A)$$

i.e.,

$$\Gamma\sigma(B) = \Gamma\sigma(A)$$

Thus,

$$\Gamma\sigma(B) = \Gamma\sigma(A)$$

□

# 5 Conclusion

This study has analyzed the exponential and approximate point spectrum of a bounded linear operator in Banach spaces. We have presented new techniques for approximating spectra of linear operators on separable Hilbert space through exposition of the operator $e^T$ where $T$ is a bounded operator in a Banach space $X$. We have defined the expression $e^T$ and the condition under which the operator $e^T$ is invertible. We also stated the Fuglede's Theorem and its proof and discussed the relationship between the boundary of the spectrum of $T$ and the approximate spectrum of $T$. Indeed, the approximate point spectrum of a bounded operator $T$ is closed and the boundary spectrum of a bounded operator $A$ is a subset of its approximate point spectrum.

# Competing Interests

Authors have declared that no competing interests exist.

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