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Investigating Computational Generalization Of Mixed Polynomial Exponential On Diophantine Equations: $\alpha^n + \beta^n + \alpha(\alpha^s \psi \beta^s)^m + D = r(u^k + v^k + w^k)$ with consecutive α and β

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ABSTRACT

Let $a, \alpha, \beta, r, u, v, w,$ and D be integers, and let $n, m, s,$ and k be non-negative exponents. This paper investigates the Diophantine equation $\alpha^n + \beta^n + a(\alpha^s \pm \beta^s)^m + D = r(u^k + v^k + w^k)$ for integer solutions and explores associated polynomial identities. Additionally, several conjectures are formulated in relation to this equation, aiming to extend the understanding of its integer solutions and structural properties. With a focus on finding integer solutions and examining underlying polynomial identities. The structure of this equation combining additive powers, multiplicative transformations, and polynomial expressions in multiple integer variables—presents unique challenges and opportunities in the search for integer solutions. By exploring cases for specific values of $n, m, s,$ and $k,$ we develop insights into solution sets, identify relationships among terms, and analyze symmetry properties inherent in this equation.

Keywords: Sequences, Diophantine equation, Integers, Polynomial identities, Factorization

1 Introduction

This study investigates the computational extension and generalization of mixed polynomial-exponential Diophantine equations, [Fathi et al., 2012, Bombieri and Bourgain, 2015, Cavallo, 2019, Simatwo, 2024] specifically examining the form $\alpha^n + \beta^n + a(\alpha^s \pm \beta^s)^m + D = r(u^k + v^k + w^k)$, where $a, \alpha, \beta, r, u, v, w$, and D are integers, with n, m, s, and k as non-negative integer exponents. The exploration focuses on solutions where α and β are consecutive integers, a condition that introduces additional structure and complexity to the equation. [Christopher, 2016, Kouropoulos, 2021, Mude, 2022, Mude et al., 2024, Lao et al., 2023] .This generalized Diophantine equation encompasses both polynomial and exponential terms, combining additive and multiplicative operations on variables raised to powers. This mix of algebraic forms not only extends classical results in number theory but also poses significant computational challenges. We explore whether specific configurations and values of the parameters n, m, s, and k yield integer solutions and seek to identify identities and relationships inherent in the equation [Najman, 2010b, Cai, 2016, Mude, 2024, Tignol, 2015]. The study also formulates conjectures regarding the solvability of the equation under various parameter constraints, as well as possible conditions for the existence of integer solutions when α and β are consecutive. These conjectures aim to deepen the theoretical understanding of mixed polynomial-exponential Diophantine equations, laying a foundation for future computational and theoretical work in this area, [Obiero and Simatwo, 2025, Mochimaru, 2005, Najman, 2010a, Osogo and Simatwo,

2 Main Results

Theorem 1.1: Consider equation satisfying condition $(a, m, n, k, r, s, D) = (1, 4, 4, 2, 2, 1, 2)$. Then the diophantine equation $\alpha^4 + \beta^4 + \alpha(\alpha^s \psi \beta^s)^4 + D = r(u_2^2 + v_2^2 + w_2^2)$ has solution in integers if α and β are consecutive.

Proof: Suppose that α and β are consecutive integers and consider the equation $\alpha^4 + \beta^4 + \alpha(\alpha^s \psi \beta^s)^4 + D = r(u_2^2 + v_2^2 + w_2^2)$

The Left Hand Side expressed as $\alpha^4 + \beta^4 + (\beta - \alpha)^4) + 2 = \alpha^4 + (\alpha + 1)^4 + 3$

Simplifies to: $2\alpha^4 + 4\alpha^3 + 6\alpha^2 + 4\alpha + 4$. Then rewriting the equation $2\alpha^4 + 4\alpha^3 + 6\alpha^2 + 4\alpha + 4 = 2(u_2^2 + v_2^2 + w_2^2)$ and dividing both sides by 2 we get: $\alpha^4 + 2\alpha^3 + 3\alpha^2 + 2\alpha + 2 = (u_2^2 + v_2^2 + w_2^2)$

To determine the value of u_2, v_2 and w_2 assume that: $u_2 = a\alpha^2 + b\alpha + c$, $v_2 = d\alpha^2 + e\alpha + f$ and $w_2 = g\alpha^2 + h\alpha + i$
Thus $u_2^2 + v_2^2 + w_2^2 = u_2 = (a\alpha^2 + b\alpha + c)^2 + (d\alpha^2 + e\alpha + f)^2 + (g\alpha^2 + h\alpha + i)^2$

$= a^2\alpha^4 + 2ab\alpha^3 + (2ac + b^2)\alpha^2 + 2bc\alpha + c^2 + d^2\alpha^4 + 2de\alpha^3 + (2df + e^2)\alpha^2 + 2ef\alpha + f^2 + g^2\alpha^4 + 2gh\alpha^3 + (2gi + h^2)\alpha^2 + 2hi\alpha + i^2$

$= (a^2 + d^2 + g^2)\alpha^4 + (2ab + 2de + 2gh)\alpha^3 + (2ac + b^2 + 2df + e^2 + 2gi + h^2)\alpha^2 + (2bc + 2ef + 2hi)\alpha + (c^2 + f^2 + i^2)$
$= \alpha^4 + 2\alpha^3 + 3\alpha^2 + 2\alpha + 1$
Matching the coefficient we have:

$$\begin{cases} a^2+d^2+g^2=1...(i) \\ 2ab+2de+2gh=2...(ii) \\ 2ac+b^2+2df+e^2+2gi+h^2=3...(iii) \\ 2bc+2ef+2hi=2...(iv) \\ c^2+f^2+i^2...(v) \end{cases}$$

Clearly, the system has nine variablea and five equations, thus there is no variable method of inspection. To solve the system we find the possible integer values. $a, b, c, d, e, f, g, h, i$ that satisfy the system. We shall use step by step approach to determine correctly the solution set from equation (i), $a^2+d^2+g^2=1$.

Assume $a=0$, thus $d^2+g^2=1$. The integer values are $d=0$ and $g=1$. Hence $a^2+d^2+g^2=0^2+0^2+1^2=1$. Substituting the solution set $a, d, g=(0,0,1)$ into equation (ii) we obtain: $2gh=2$

Dividing both sides by two we obtain $gh=1$, clearly, $g=1$ and $h=1$. Thus $(g,h)=(1,1)$ is a solution.

Substituting the solution $(a,d,g,h,e,i)(0,0,1,0,1,0,0)$ into equation (iv), we have $c=1$. Finally:
Finally, considering equation (v) we have $c^2+f^2+i^2=1^2+1^2+0^2=2$, which satisfies the equation consequently:

$$u_2=0$$
$$v_2=1$$
$$w_2=\alpha^2+\alpha+1$$

α^4	β^4	$(\alpha-\beta)^4+2$	I_2	$U_2^2=0^2$	$v_2^2=I^2$	$w_2^2=\alpha^2+\alpha+1$
1	16	3	20	0	1	9
16	81	3	100	0	1	49
81	256	3	840	0	1	169
256	625	3	884	0	1	441
625	1296	3	1924	0	1	961
1296	2401	3	3700	0	1	1849
2401	4096	3	6500	0	1	3249

Thus upon I_2 we get:

α^4	β^4	$(\alpha-\beta)^4+2$	I_2	$U_2^2=0^2$	$v_2^2=I^2$	$w_2^2=\alpha^2+\alpha+1$	I_2
1	16	3	20	0	1	9	20
16	81	3	100	0	1	49	100
81	256	3	840	0	1	169	840
256	625	3	884	0	1	441	884
625	1296	3	1924	0	1	961	1924
1296	2401	3	3700	0	1	1849	3700
2401	4096	3	6500	0	1	3249	6500

3 Conclusion

To sum up, this research has provided integral solution for the diophantine equation $\alpha^n + \beta^n + \alpha(\alpha^s \psi \beta^s)^m + D = r(u^k + v^k + w^k)$ where α and β are consecutive integers thus giving an investigation on the same families of the diophantine equation with different exponent and the difference between alpha and beta greater or equal to 2.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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Competing Interests

Author has declared that no competing interests exist.

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