Vol.2 No. 1(2025),pp.117-131, African Scientific Annual Review www.asarev.net ISSN:3007-0902

Analyzing Dunford Property for Operators Satisfying $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r}$

Wanjala Victor [1*]
Beatrice Adhiambo Obiero[2]

[1*] wanjalavictor421@gmail.com

[1]Department of Mathematics and Physical Sciences, Maasai Mara University, P.O Box 861-20500, Narok-Kenya
[2]Department of Mathematics and Computing, Rongo University, P.O Box 103-40404, Rongo-Kenya.

Original Research Article
DOI: https://doi.org/10.51867/asarev.maths.2.1.8

Abstract

In this study, we introduce a new class of operators defined by the properties $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r}$ and $\mathfrak{T}^{\frac{r}{2}}\mathfrak{S}^q\mathfrak{T}^{\frac{r}{2}} = \mathfrak{T}^{2r}$, for integers $r > q \geq 0$. Our main objective is to investigate the Dunford property, commonly referred to as property $(\mathcal{C})$, for the operators $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q$ and $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$, under the condition that $\mathfrak{S}^{2r} \in \mathfrak{B}(\mathfrak{X})$. This research expands the framework of operator theory by introducing new operator classes through operator identities and extending existing ones. The motivation stems from the central role of operator equations in functional analysis and operator theory, where many fundamental problems in mathematics and physics can be reformulated as operator equations, yet certain classes remain insufficiently explored. Our methodology involves an iterative analysis of local spectral subspaces and their interactions under the given operator identities. The results demonstrate that the introduced classes of operators satisfy the single-valued extension property (SVEP) and possess property (Q). Moreover, we establish that if $\mathfrak{S}^{\frac{3r}{2}}$ has property $(\mathcal{C})$, then both $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q$ and $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ inherit this property. These findings enrich the theory with broader generalizations and open avenues for further exploration of spectral properties and applications in mathematical and scientific contexts.

Keyword: SVEP property, Dunford's property $(\mathcal{C})$, Local spectral theory.

1 Introduction

This research introduces a novel class of operators defined by their adherence to property (C), characterized by the conditions

$$\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r} \quad \text{and} \quad \mathfrak{T}^{\frac{r}{2}}\mathfrak{S}^q\mathfrak{T}^{\frac{r}{2}} = \mathfrak{T}^{2r}, \quad r > q \geq 0;$$

such that r and q can either be even or positive intergers .

The motivation for considering these relations lies in their unifying and extending role within operator theory. The appearance of fractional powers $(r/2)$ reflects a natural extension of well-known operator identities involving integer powers, while the parameter q introduces flexibility that captures intermediate behaviors between classical cases. In particular, when $q = 0$ or $r = q + 1$, the equations reduce to familiar forms related to established operator classes, thereby demonstrating that this framework subsumes earlier results while going beyond them. By investigating this generalization, we obtain a broader perspective on operator equations that govern local spectral behavior. This motivates the study of their Dunford property and related structural aspects, highlighting the significance of fractional and mixed powers in revealing operator behaviors not captured by the classical setting.

Local spectral properties have various applications, including in functional analysis, operator equations, and mathematical physics. They also find applications in the study of soft graphs, where the adjacency operator of a soft graph can be analyzed using local spectral theory. In particular, the local spectrum provides information about the stability, connectivity, and clustering properties of soft graphs, thereby linking operator-theoretic methods with soft set based graph models. Some interesting results related to soft graphs are presented by Jinta *et al.* [1, 2].

The study of operator equations has been explored by various researchers, covering aspects from basic properties to local spectral characteristics. In this context, we highlight several works that discuss standard spectral properties, such as Bishop's property, including those by Shen [3], Laursen and Neumann [4], Salah [5], and Hadji and Zguitti [6], among others.

For instance, Salah [5] analyzed Bishop's property in the class of Helton operators, providing criteria for a 2×2 matrix to satisfy Bishop's property. Additionally, he examined conditions under which the sum $\mathfrak{S} + \mathfrak{T}$ retains Bishop's property for bounded linear operators $\mathfrak{S}$ and $\mathfrak{T}$.

Many researchers have explored the local spectral properties of various classes of operators. For example, the local spectral properties of extended Hamiltonian operators and their adjoint operators were studied by Wurichaihu and Chen [7]. Property (bz) was examined by Elvis *et al.* [8]. For recent developments concerning local spectral properties of different types of operators, we refer the reader to notable results presented by Elvis *et al.* [9], Mhamed and Saber [10], Macias *et al.* [11], and Bermudez *et al.* [12].

We are concerned with linear operators that satisfy certain common basic properties, particularly local spectral properties. It was shown by Barens [13] that $\lambda - \mathfrak{S}\mathfrak{R}$ and $\lambda - \mathfrak{R}\mathfrak{S}$ share many fundamental properties common to bounded operators $\mathfrak{R}$ and $\mathfrak{S}$, whenever $0 \neq \lambda$, for operators $\mathfrak{S} : \mathfrak{X} \to \mathfrak{Y}$ and $\mathfrak{R} : \mathfrak{Y} \to \mathfrak{X}$. Other basic properties, such as subdecomposability related to $\mathfrak{R}\mathfrak{S}$ and $\mathfrak{S}\mathfrak{R}$, were discussed by Lin *et al.* [14]. Local spectral properties associated with $\mathfrak{R}\mathfrak{S}$ and $\mathfrak{S}\mathfrak{R}$ were analyzed by Benhida and Zerouali [15] and by Yoo [16].

Local spectral properties of multipliers were studied by Aiena [17]. The Dunford property for the operators $\mathfrak{R}\mathfrak{S}$ and $\mathfrak{S}\mathfrak{R}$ was later examined by Aiena and Gonzalez [18], where it was demonstrated that both $\mathfrak{R}\mathfrak{S}$ and $\mathfrak{S}\mathfrak{R}$ possess Dunford's property (C). It was also shown that $\mathfrak{R}$ itself satisfies property (C), and that $\mathfrak{S}$, $\mathfrak{R}\mathfrak{S}$, and $\mathfrak{S}\mathfrak{R}$ belong to $\mathfrak{B}(\mathfrak{X})$ in the case where $\mathfrak{R}$ and $\mathfrak{S}$ satisfy the operator equations $\mathfrak{R}\mathfrak{S}\mathfrak{R} = \mathfrak{R}^2$ and $\mathfrak{S}\mathfrak{R}\mathfrak{S} = \mathfrak{S}^2$.

Benhida and Zerouali [19] further analyzed the spectral characteristics of the operators $\mathfrak{R}\mathfrak{S}$ and $\mathfrak{S}\mathfrak{R}$ and explored their applications to Aluthge transformations and upper triangular operator matrices.

Schmoeger [20] presented results on the Drazin invertibility of operator equations related to this study, and later on provided common spectral properties, Schmoeger [21]. Some literature covering linear operators satisfying the equation $\mathfrak{A}\mathfrak{S}\mathfrak{A} = \mathfrak{A}\mathfrak{T}\mathfrak{A}$ includes the work of Zeng and Zhong [22, 23], where they established conditions under which $\mathfrak{A}\mathfrak{S}\mathfrak{A} = \mathfrak{A}\mathfrak{T}\mathfrak{A}$ share common regularities for the operators $\mathfrak{A}\mathfrak{T}$ and $\mathfrak{S}\mathfrak{A}$. Subsequently, Tajmouati *et al.* [24] analyzed the local spectral properties of operators satisfying $\mathfrak{B}\mathfrak{S}\mathfrak{A} = \mathfrak{A}\mathfrak{T}\mathfrak{B}$.

Operators satisfying the relations $\mathfrak{A}\mathfrak{B}\mathfrak{A} = \mathfrak{A}^2$ and $\mathfrak{B}\mathfrak{A}\mathfrak{B} = \mathfrak{B}^2$ for bounded linear operators $\mathfrak{A}$ and $\mathfrak{B}$ have been studied extensively (see Duggal [25] for further references). This class was later generalized by Triolo [26] to operators satisfying $\mathfrak{R}^n\mathfrak{S}\mathfrak{R}^n = \mathfrak{R}^j$ for integers $j \geq n \geq 0$, where $\mathfrak{R}$ and $\mathfrak{S}$ are bounded operators. While the class defined by $\mathfrak{A}\mathfrak{B}\mathfrak{A} = \mathfrak{A}^2$ has been well studied [25, 26], the generalization to fractional powers, as in

$$\mathfrak{A}^{\frac{r}{2}}\mathfrak{B}^q\mathfrak{A}^{\frac{r}{2}} = \mathfrak{A}^{2r},$$

remains unexplored and presents new challenges due to the interplay between fractional powers of one operator and integer powers of another.

The intuition behind these defining relations is that they extend classical operator equations by introducing a balance between fractional and integer powers. For instance, in the simplest case when $r = 2$ and $q = 1$, the first relation reduces to

$$\mathfrak{S}\,\mathfrak{T}\,\mathfrak{S} = \mathfrak{S}^4,$$

which can be viewed as an extension of the familiar form $\mathfrak{A}\mathfrak{B}\mathfrak{A} = \mathfrak{A}^2$ to higher-order interactions. This illustrates how the parameter q interpolates between classical results and new operator behaviors.

The present paper makes three main contributions. First, we introduce the new class $\mathfrak{S}(r/2, 2r)$, which extends the families of operators previously studied by Aiena and Gonzalez [18], Duggal [25], and Triolo [26]. Second, we provide a detailed structural analysis of these operators, with particular emphasis on their local spectral properties and their connections to the Dunford property. Finally, we demonstrate that the newly defined class is independent from existing families of operators, thereby enlarging the theoretical framework of operator equations in Hilbert space theory.

Throughout this work, $\mathfrak{X}$ denotes a Banach space over the complex field $\mathbb{C}$, while $\mathfrak{B}(\mathfrak{X})$ denotes the set of bounded linear operators on $\mathfrak{X}$.

Let $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$. Then $\sigma(\mathfrak{T})$ denotes the spectrum of $\mathfrak{T}$, $\mathfrak{T}^*$ denotes the adjoint of $\mathfrak{T}$, and $\rho_{\mathfrak{T}}(x)$ denotes the local resolvent set of $\mathfrak{T}$ at a vector $x \in \mathfrak{X}$. The notation $\mathfrak{X}_{\mathfrak{T}}(\mathfrak{F})$ refers to the local spectral subspace corresponding to a closed subset $\mathfrak{F} \subseteq \mathbb{C}$. Finally, $\sigma_{\mathfrak{T}}(x)$ represents the local spectrum of $\mathfrak{T}$ at the point $x \in \mathfrak{X}$.

Definition 1.1. *An operator $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ is said to be upper semi-Fredholm, denoted by $\mathfrak{T} \in \phi_+(\mathfrak{X})$, if $\mathfrak{T}(\mathfrak{X})$ is closed and the kernel of $\mathfrak{T}$ is finite-dimensional. It is said to be lower semi-Fredholm, denoted by $\mathfrak{T} \in \phi_-(\mathfrak{X})$, if the range $\mathfrak{T}(\mathfrak{X})$ has finite codimension.*

Definition 1.2. *For a linear operator $\mathfrak{T}$ on a vector space $\mathfrak{X}$, the hyperrange of $\mathfrak{T}$ is defined as*

$$\mathfrak{T}^{\infty}(\mathfrak{X}) := \bigcap_{n\in\mathbb{N}} \mathfrak{T}^n(\mathfrak{X}).$$

Generally, $\mathfrak{T}(\mathfrak{T}^{\infty}(\mathfrak{X})) \subseteq \mathfrak{T}^{\infty}(\mathfrak{X})$. We seek conditions for $\mathfrak{T}(\mathfrak{T}^{\infty}(\mathfrak{X})) = \mathfrak{T}^{\infty}(\mathfrak{X})$. For a linear operator $\mathfrak{T}$ on $\mathfrak{X}$, the sequences are:

$$\{0\} = \ker(\mathfrak{T}^0) \subseteq \ker(\mathfrak{T}) \subseteq \ker(\mathfrak{T}^2) \subseteq \dots$$

$$\mathfrak{X} = \mathfrak{T}^0(\mathfrak{X}) \supseteq \mathfrak{T}(\mathfrak{X}) \supseteq \mathfrak{T}^2(\mathfrak{X}) \supseteq \dots$$

Definition 1.3. *The* analytic core *of the operator $\mu\mathfrak{I} - \mathfrak{T}$, denoted by $\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{T})$, is the set*

$$\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{T}) := \{x \in \mathfrak{X} : \mu \notin \sigma_{\mathfrak{T}}(x)\}.$$

Definition 1.4. $\mathfrak{X}_{\mathfrak{T}}(\mathfrak{F}) := \{x \in \mathfrak{X} : \sigma_{\mathfrak{T}}(x) \subseteq \mathfrak{F}\}$ *is the $\mathfrak{T}$-invariant local spectral subspace of $\mathfrak{X}$ corresponding to the set $\mathfrak{F}$, where $\mathfrak{F} \subseteq \mathbb{C}$ is closed.*

Definition 1.5. *Let $x \in \mathfrak{X}$. The* local resolvent set *$\rho_{\mathfrak{T}}(x)$ is the union of all open subsets $\mathfrak{U} \subseteq \mathbb{C}$ for which there exists an analytic function $\mathfrak{f} : \mathfrak{U} \to \mathfrak{X}$ such that*

$$(\lambda\mathfrak{I} - \mathfrak{T})\mathfrak{f}(\lambda) = x, \quad \forall\lambda \in \mathfrak{U}.$$

Definition 1.6. *An operator $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ is said to have the single-valued extension property (SVEP) ; property (β)) ; if the only analytic function $\mathfrak{f} : \mathfrak{U} \to \mathfrak{X}$ that satisfies $(\mathfrak{T} - \lambda)\mathfrak{f} : (\lambda) = 0$ for all $\lambda \in \mathfrak{U}$, where $\mathfrak{U} \subseteq \mathbb{C}$ is open, is the zero function $\mathfrak{f} :\equiv 0$.*

Definition 1.7. *An operator $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ is said to have property (C) if $\mathfrak{X}_{\mathfrak{T}}(\mathfrak{F})$ is closed for every closed set $\mathfrak{F} \subseteq \mathbb{C}$.*

Definition 1.8. *An operator $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ is said to have property (Q) if the quasi-nilpotent part*

$$\mathfrak{H}_0(\mu\mathfrak{I} - \mathfrak{T}) := \left\{ x \in \mathfrak{X} : \limsup_{n\to\infty} \|(\mu\mathfrak{I} - \mathfrak{T})^n x\|^{1/n} = 0 \right\}$$

is closed for every $\mu \in \mathbb{C}$. The following implications hold:

Property (C) $\Rightarrow$ Property (Q) $\Rightarrow$ SVEP.

Moreover, if $\mathfrak{T}$ has SVEP, then $\mathfrak{X}_{\mathfrak{T}}(\mu) = \mathfrak{H}_0(\mu\mathfrak{I} - \mathfrak{T})$.

We shall denote by $\mathfrak{S}_{(\frac{r}{2},2r)}$ a pair of operators $(\mathfrak{S}, \mathfrak{T})$ satisfying the identities

$$\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r} \quad \text{and} \quad \mathfrak{T}^{\frac{r}{2}}\mathfrak{S}^q\mathfrak{T}^{\frac{r}{2}} = \mathfrak{T}^{2r},$$

for integers $r > q \geq 0$.

We restate the following known results, which will be used in our findings.

Lemma 1.1. *[*Lemma 1.1, [27]*] Let $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ and $x \in \mathfrak{X}$. Then*

$$\sigma_{\mathfrak{T}}(\mathfrak{T}x) \subseteq \sigma_{\mathfrak{T}}(x) \subseteq \sigma_{\mathfrak{T}}(\mathfrak{T}x) \cup \{0\}.$$

If $\mathfrak{T}$ is injective, then $\sigma_{\mathfrak{T}}(\mathfrak{T}x) = \sigma_{\mathfrak{T}}(x)$.

Lemma 1.2. *[*Lemma 2.3, [18]*] Let $\mathfrak{F}$ be a closed subset of $\mathbb{C}$, and let $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$. If $0 \in \mathfrak{F}$ and $\mathfrak{T}x \in \mathfrak{X}_{\mathfrak{T}}(\mathfrak{F})$, then $x \in \mathfrak{X}_{\mathfrak{T}}(\mathfrak{F})$.*

Lemma 1.3. *[*Lemma 2.4, [18]*] Suppose $\mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ has the SVEP, and let $\mathfrak{F}$ be a closed subset of $\mathbb{C}$ such that $\mathfrak{Z} := \mathfrak{X}_{\mathfrak{T}}(\mathfrak{F})$ is closed. If $\mathfrak{A} := \mathfrak{T}|_{\mathfrak{X}_{\mathfrak{T}}(\mathfrak{F})}$, then*

$$\mathfrak{X}_{\mathfrak{T}}(\mathfrak{K}) = \mathfrak{Z}_{\mathfrak{A}}(\mathfrak{K}) \quad \text{for all closed } \mathfrak{K} \subseteq \mathfrak{F}.$$

Proposition 1.1. *[*Proposition 2.1, [15]*] Let $\lambda \in \mathbb{C}$. The operator $\mathfrak{R}\mathfrak{S}$ has the SVEP (respectively, property (β)) at λ if and only if $\mathfrak{S}\mathfrak{R}$ does. Hence, $\mathfrak{R}\mathfrak{S}$ has the SVEP (respectively, property (β)) if and only if $\mathfrak{S}\mathfrak{R}$ does.*

Vol.2 No. 1(2025),pp.117-131, African Scientific Annual Review www.asarev.net ISSN:3007-0902

Main results

Lemma 1.4. *For all $x \in \mathfrak{X}$ and integers $r > q \geq 0$, the following inclusions hold:*

$$\sigma_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{S}^{\frac{r}{2}}x) \subseteq \sigma_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(x), \tag{1}$$

and

$$\begin{aligned} \sigma_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}x) &\subseteq \sigma_{\mathfrak{S}^{\frac{3r}{2}}}(x), \\ \sigma_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}x) &\subseteq \sigma_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{S}^{\frac{r}{2}}x). \end{aligned} \tag{2}$$

Proof. The strategy is to show that if a point μ_0 belongs to the local resolvent set of one operator, then it must also belong to the local resolvent set of the other. This yields the spectral inclusions. We treat each inclusion separately.

Step 1: Proof of (1). Assume $\mu_0 \in \rho_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(x)$. By definition, there exists an open neighborhood $\mathfrak{U}_0$ of μ_0 and an analytic function $\mathfrak{f} : \mathfrak{U}_0 \to \mathfrak{X}$ such that

$$(\mu\mathfrak{I} - \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})\mathfrak{f}(\mu) = x, \quad \mu \in \mathfrak{U}_0. \tag{3}$$

Applying $\mathfrak{S}^{\frac{r}{2}}$ to both sides of (3), we obtain

$$\mathfrak{S}^{\frac{r}{2}}x = \mathfrak{S}^{\frac{r}{2}}(\mu\mathfrak{I} - \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})\mathfrak{f}(\mu) = (\mu\mathfrak{S}^{\frac{r}{2}} - \mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})\mathfrak{f}(\mu).$$

Since the defining relation gives $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r}$, we rewrite:

$$\mathfrak{S}^{\frac{r}{2}}x = (\mu\mathfrak{S}^{\frac{r}{2}} - \mathfrak{S}^{2r})\mathfrak{f}(\mu) = (\mu\mathfrak{I} - \mathfrak{S}^{\frac{3r}{2}})\big(\mathfrak{S}^{\frac{r}{2}}\mathfrak{f}(\mu)\big).$$

Now, since $\mathfrak{f}(\mu)$ is analytic and $\mathfrak{S}^{\frac{r}{2}}$ is a bounded linear operator, the function $\mu \mapsto \mathfrak{S}^{\frac{r}{2}}\mathfrak{f}(\mu)$ is also analytic. Hence $\mu_0 \in \rho_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{S}^{\frac{r}{2}}x)$, proving the inclusion in (1).

Step 2: First inclusion in (2). Suppose $\mu_0 \in \rho_{\mathfrak{S}^{\frac{3r}{2}}}(x)$. Then there exists an open neighborhood $\mathfrak{U}_0$ of μ_0 and an analytic function $\mathfrak{f} : \mathfrak{U}_0 \to \mathfrak{X}$ such that

$$(\mu\mathfrak{I} - \mathfrak{S}^{\frac{3r}{2}})\mathfrak{f}(\mu) = x, \quad \mu \in \mathfrak{U}_0.$$

Applying $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ to both sides gives

$$\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}x = (\mu\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} - \mathfrak{T}^q\mathfrak{S}^{2r})\mathfrak{f}(\mu).$$

Using the relation $\mathfrak{T}^q\mathfrak{S}^{2r} = \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$, this simplifies to

$$\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}x = (\mu\mathfrak{I} - \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})\big(\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}\mathfrak{f}(\mu)\big).$$

Since the map $\mu \mapsto \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}\mathfrak{f}(\mu)$ is analytic, we deduce that $\mu_0 \in \rho_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}x)$. This proves the first inclusion in (2).

Step 3: Second inclusion in (2). Assume $\mu_0 \in \rho_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{S}^{\frac{r}{2}}x)$. Then there exists $\mathfrak{U}_0$ and analytic $\mathfrak{f} : \mathfrak{U}_0 \to \mathfrak{X}$ such that

$$(\mu\mathfrak{I} - \mathfrak{S}^{\frac{3r}{2}})\mathfrak{f}(\mu) = \mathfrak{S}^{\frac{r}{2}}x, \quad \mu \in \mathfrak{U}_0.$$

Repeating the same reasoning as in Step 2, but with $\mathfrak{S}^{\frac{r}{2}}x$ in place of x, we conclude that $\mu_0 \in \rho_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}x)$.

This completes the proof. □

Theorem 1.1. *Let* $\mathfrak{F}$ *be a subspace of* $\mathbb{C}$ *with* $0 \in \mathfrak{F}$. *Then*

$$\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F}) = \mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F}).$$

Proof. We divide the proof into two parts.

(1) Inclusion $\mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F}) \subseteq \mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F})$.

Let $\{x_j\}$ be a sequence in $\mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F})$ such that $x_j \to x$ in $\mathfrak{X}$. Then, by definition,

$$\sigma_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(x_j) \subseteq \mathfrak{F}, \quad \forall j \in \mathbb{N}.$$

From equation (1) we know that

$$\sigma_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{S}^{\frac{r}{2}}x_j) \subseteq \sigma_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(x_j) \subseteq \mathfrak{F}.$$

Now apply Lemma 1.1: since $0 \in \mathfrak{F}$,

$$\sigma_{\mathfrak{S}^{\frac{3r}{2}}}(x_j) \subseteq \sigma_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{S}^{\frac{r}{2}}x_j) \cup \{0\} \subseteq \mathfrak{F}.$$

Thus $x_j \in \mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F})$ for all j. Since $\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F})$ is closed, the limit point x also lies in it. Hence

$$\mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F}) \subseteq \mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F}).$$

(2) Inclusion $\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F}) \subseteq \mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F})$.

Let $\{x_j\}$ be a sequence in $\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F})$ with $x_j \to x$. Then

$$\sigma_{\mathfrak{S}^{\frac{3r}{2}}}(x_j) \subseteq \mathfrak{F}, \quad \forall j \in \mathbb{N}.$$

From equation (2) we obtain

$$\sigma_{\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}}(\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}} x_j) \subseteq \sigma_{\mathfrak{S}^{\frac{3r}{2}}}(x_j) \subseteq \mathfrak{F}.$$

Thus $\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}} x_j \in \mathfrak{X}_{\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F})$. By Lemma 1.2, this implies $x_j \in \mathfrak{X}_{\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F})$. Passing to the limit, we conclude $x \in \mathfrak{X}_{\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F})$. Hence

$$\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F}) \subseteq \mathfrak{X}_{\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F}).$$

Combining (1) and (2) gives the desired equality:

$$\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F}) = \mathfrak{X}_{\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F}).$$

□

Lemma 1.5. *Let $\mathfrak{S}, \mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ be such that $\mathfrak{S}, \mathfrak{T} \in \mathfrak{S}_{(\frac{r}{2}, 2r)}$ for integers $r > q \geq 0$. If $\mathfrak{S}^{\frac{3r}{2}}$ has the SVEP, then both $\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}$ and $\mathfrak{S}^{\frac{r}{2}} \mathfrak{T}^q$ have the SVEP.*

Proof. From Proposition 1.1, $\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}$ possesses the SVEP if and only if $\mathfrak{S}^{\frac{r}{2}} \mathfrak{T}^q$ does. Suppose $\mathfrak{S}^{\frac{3r}{2}}$ has the SVEP at μ_0, and let $\mathfrak{f} : \mathfrak{U}_0 \to \mathfrak{X}$ be an analytic function on a neighborhood $\mathfrak{U}_0$ of μ_0 such that

$$(\mu \mathfrak{I} - \mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}) \mathfrak{f}(\mu) = 0 \quad \text{for all } \mu \in \mathfrak{U}_0.$$

Then,

$$\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}} \mathfrak{f}(\mu) = \mu \mathfrak{f}(\mu).$$

Applying $\mathfrak{S}^{\frac{r}{2}}$ gives:

$$\mathfrak{S}^{\frac{r}{2}}(\mu \mathfrak{I} - \mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}) \mathfrak{f}(\mu) = (\mu \mathfrak{S}^{\frac{r}{2}} - \mathfrak{S}^{2r}) \mathfrak{f}(\mu) = (\mu \mathfrak{I} - \mathfrak{S}^{\frac{3r}{2}}) \mathfrak{S}^{\frac{r}{2}} \mathfrak{f}(\mu) = 0.$$

Since $\mathfrak{S}^{\frac{3r}{2}}$ has the SVEP at μ_0, it follows that $\mathfrak{S}^{\frac{r}{2}} \mathfrak{f}(\mu) = 0$, and thus,

$$\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}} \mathfrak{f}(\mu) = \mu \mathfrak{f}(\mu) = 0.$$

As $0 \notin \mathfrak{U}_0$, we conclude that $\mathfrak{f}(\mu) = 0$ for all $\mu \in \mathfrak{U}_0$, and by continuity, $\mathfrak{f}(0) = 0$. Therefore, $\mathfrak{T}^q \mathfrak{S}^{\frac{r}{2}}$ has the SVEP at μ_0. □

Remark 1.1. *A case where $0 \notin \mathfrak{F}$ is considered in the next result.*

Theorem 1.2. *Let $\mathfrak{F}$ be a closed subset of $\mathbb{C}$ such that $0 \notin \mathfrak{F}$. Suppose $\mathfrak{S}, \mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ with $\mathfrak{S}, \mathfrak{T} \in \mathfrak{S}_{(\frac{r}{2},2r)}$ for integers $r > q \geq 0$. If $\mathfrak{S}^{\frac{3r}{2}}$ has the SVEP and $\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F})$ is closed, then $\mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F})$ is also closed.*

Proof. Define $\mathfrak{F}_\alpha := \mathfrak{F} \cup \{0\}$. Then $\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F}_\alpha)$ is closed by assumption. By Theorem 1.1, $\mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F}_\alpha)$ is also closed. Since Lemma 1.5 guarantees that $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ has the SVEP, Lemma 1.3 implies that $\mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F})$ is closed. □

Theorem 1.3. *Let $\mathfrak{S}, \mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ be such that $\mathfrak{S}, \mathfrak{T} \in \mathfrak{S}_{(\frac{r}{2},2r)}$ for integers $r > q \geq 0$. If $\mathfrak{S}^{\frac{3r}{2}}$ has property (C), then both $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ and $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q$ have property (C).*

Proof. Let $\mathfrak{F}$ be a subspace and suppose $\mathfrak{S}^{\frac{3r}{2}}$ has Dunford's property (C). Then, $\mathfrak{S}^{\frac{3r}{2}}$ has the SVEP. If $0 \in \mathfrak{F}$, then from Theorem 1.1, $\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{F})$ is a subspace, and hence $\mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F})$ is a subspace as well. If $0 \notin \mathfrak{F}$, then Theorem 1.2 ensures that $\mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{F} \cup \{0\})$ is a subspace. Therefore, $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ has Dunford's property (C). □

Theorem 1.4. *Let $\mathfrak{S}, \mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ with $\mathfrak{S}, \mathfrak{T} \in \mathfrak{S}_{(\frac{r}{2},2r)}$ for integers $r > q \geq 0$. If $\mathfrak{S}^{\frac{3r}{2}}$ satisfies property (Q), then $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ also satisfies property (Q).*

Proof. Assume $\mathfrak{S}^{\frac{3r}{2}}$ has property (Q). Then, $\mathfrak{S}^{\frac{3r}{2}}$ has the SVEP, and from Lemma 1.5, $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ also has the SVEP. By Definition 1.8 and assumption, we have $\mathfrak{H}_0(\mu\mathfrak{I} - \mathfrak{S}^{\frac{3r}{2}}) = \mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\{\mu\})$ is closed for all $\mu \in \mathbb{C}$. From Definition 1.8 and Theorem 1.1, it follows that $\mathfrak{H}_0(\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}) = \mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\{0\})$ is also closed.

Now let $0 \neq \mu \in \mathbb{C}$. From Proposition 3.3.1 in [4], we obtain:

$$\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\{\mu\} \cup \{0\}) = \mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\{\mu\}) + \mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\{0\}) = \mathfrak{H}(\mu\mathfrak{I} - \mathfrak{S}^{\frac{3r}{2}}) + \mathfrak{H}_0(\mathfrak{S}^{\frac{3r}{2}}).$$

Since $\mathfrak{S}^{\frac{3r}{2}}$ is upper semi-Fredholm, SVEP at 0 implies that $\mathfrak{H}_0(\mathfrak{S}^{\frac{3r}{2}})$ is finite-dimensional by Theorem 3.18 in [28]. Hence, $\mathfrak{X}_{\mathfrak{S}^{\frac{3r}{2}}}(\{\mu\} \cup \{0\})$ is closed. Then by Theorem 1.3, we conclude that $\mathfrak{H}_0(\mu\mathfrak{I} - \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}) = \mathfrak{X}_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\{\mu\})$ is closed. Therefore, $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ has property (Q). □

Theorem 1.5. *Let $\mathfrak{S}, \mathfrak{T} \in \mathfrak{B}(\mathfrak{X})$ be such that*

$$\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r}, \quad r > q \geq 0.$$

(i) If $\mu \neq 0$ for $\mu \in \mathbb{C}$, then $\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{S}^{\frac{3r}{2}})$ is closed if and only if $\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})$ is closed, or equivalently, $\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q)$ is closed.

(ii) If $\mathfrak{S}^{\frac{3r}{2}}$ is injective, then $\mathfrak{K}(\mu\mathfrak{I}-\mathfrak{S}^{\frac{3r}{2}})$ is closed if and only if $\mathfrak{K}(\mu\mathfrak{I}-\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})$ is closed, or equivalently, $\mathfrak{K}(\mu\mathfrak{I}-\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q)$ is closed for every $\mu\in\mathbb{C}$.

Proof. Step 1: Intertwining relations. From the assumption we have

$$\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}=\mathfrak{S}^{2r}.$$

Multiplying on the left by $\mathfrak{S}^{\frac{r}{2}}$ gives

$$\mathfrak{S}^{\frac{r}{2}}\big(\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}\big)=\mathfrak{S}^{\frac{3r}{2}}.$$

Multiplying on the right by $\mathfrak{S}^{\frac{r}{2}}$ gives

$$\big(\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\big)\mathfrak{S}^{\frac{r}{2}}=\mathfrak{S}^{\frac{3r}{2}}.$$

Hence $\mathfrak{S}^{\frac{3r}{2}}$, $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$, and $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q$ are intertwined by $\mathfrak{S}^{\frac{r}{2}}$.

Step 2: Case $\mu\neq 0$. For any sequence $\{x_n\}\subset\mathfrak{K}(\mu\mathfrak{I}-\mathfrak{S}^{\frac{3r}{2}})$,

$$(\mu\mathfrak{I}-\mathfrak{S}^{\frac{3r}{2}})x_n\to 0.$$

Using the intertwining identity

$$\mathfrak{S}^{\frac{r}{2}}(\mu\mathfrak{I}-\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})=(\mu\mathfrak{I}-\mathfrak{S}^{\frac{3r}{2}})\mathfrak{S}^{\frac{r}{2}},$$

we see that if $\mathfrak{K}(\mu\mathfrak{I}-\mathfrak{S}^{\frac{3r}{2}})$ is closed, then so is $\mathfrak{K}(\mu\mathfrak{I}-\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})$. By symmetry, the reverse implication also holds via

$$(\mu\mathfrak{I}-\mathfrak{S}^{\frac{3r}{2}})\mathfrak{S}^{\frac{r}{2}}=\mathfrak{S}^{\frac{r}{2}}(\mu\mathfrak{I}-\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q).$$

Thus the equivalence in (i) holds.

Step 3: Case $\mu=0$. If $\mu=0$, we compare $\mathfrak{K}(-\mathfrak{S}^{\frac{3r}{2}})$ and $\mathfrak{K}(-\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})$. If $\mathfrak{S}^{\frac{3r}{2}}$ is injective, then $\ker(\mathfrak{S}^{\frac{3r}{2}})=\{0\}$. The intertwining relations from Step 1 imply that $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ and $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q$ are also injective, so their analytic cores coincide. Therefore the equivalence in (ii) extends to all $\mu\in\mathbb{C}$, including $\mu=0$. □

Combining Theorem 1.5 with Corollary 1 in [26], we obtain the following result.

Corollary 1.1. *Suppose $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}=\mathfrak{S}^{2r}$ and $\mathfrak{T}^{2r}\mathfrak{S}^q\mathfrak{T}^{2r}=\mathfrak{T}^{\frac{r}{2}}$ for integers $r>q\geq 0$ and $\mu\neq 0$. Then the following are equivalent:*

(i) $\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{S}^{2r})$ *is a subspace.*

(ii) $\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}})$ *is a subspace.*

(iii) $\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q)$ *is a subspace.*

(iv) $\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{T}^{\frac{r}{2}})$ *is a subspace.*

If $\mathfrak{S}$ is injective, the equivalence holds for all $\mu \in \mathbb{C}$ including $\mu = 0$.

Proof. We start from the assumptions

$$\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r}, \qquad \mathfrak{T}^{2r}\mathfrak{S}^q\mathfrak{T}^{2r} = \mathfrak{T}^{\frac{r}{2}}.$$

These relations imply that powers of $\mathfrak{S}$ and $\mathfrak{T}$ are intertwined through fractional powers and iterates, so that analytic core subspaces associated to $\mathfrak{S}^{2r}$ and $\mathfrak{T}^{\frac{r}{2}}$ transfer equivalently to those of $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ and $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q$.

Step 1. (i) ⇔ (ii). If $\mu \neq 0$, Theorem 1.5(i) applied to $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r}$ shows that

$$\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{S}^{2r}) \text{ is closed} \quad \Longleftrightarrow \quad \mathfrak{K}(\mu\mathfrak{I} - \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}) \text{ is closed.}$$

Step 2. (ii) ⇔ (iii). By symmetry, the intertwining relation also yields

$$\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}} = \mathfrak{S}^{2r} \quad \Longrightarrow \quad \mathfrak{K}(\mu\mathfrak{I} - \mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}) \Longleftrightarrow \mathfrak{K}(\mu\mathfrak{I} - \mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q).$$

Step 3. (iii) ⇔ (iv). Using the second condition $\mathfrak{T}^{2r}\mathfrak{S}^q\mathfrak{T}^{2r} = \mathfrak{T}^{\frac{r}{2}}$, an identical argument to Step 1 applied with $\mathfrak{T}$ in place of $\mathfrak{S}$ shows that

$$\mathfrak{K}(\mu\mathfrak{I} - \mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q) \Longleftrightarrow \mathfrak{K}(\mu\mathfrak{I} - \mathfrak{T}^{\frac{r}{2}}).$$

Step 4. Injectivity case. If $\mathfrak{S}$ is injective, then $\mathfrak{S}^{\frac{3r}{2}}$ is injective. By Theorem 1.5(ii), the equivalence of (i)–(iv) extends to $\mu = 0$ as well.

Combining Steps 1–3 proves that statements (i)–(iv) are equivalent for all $\mu \neq 0$, and for all $\mu \in \mathbb{C}$ when $\mathfrak{S}$ is injective. □

Conclusion

In this work, we have introduced and analyzed a new class of operators satisfying conditions such as the single-valued extension property (SVEP), property (Q), and related local spectral conditions, thereby extending the existing framework of operator theory. A central contribution of this study is the demonstration of key spectral inclusions: for all $x \in \mathfrak{X}$ and integers $r > q \geq 0$, we established that

$$\sigma_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{S}^{\frac{r}{2}}x) \subseteq \sigma_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(x),$$

and

$$\sigma_{\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}}(\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}x) \subseteq \sigma_{\mathfrak{S}^{\frac{3r}{2}}}(x) \cap \sigma_{\mathfrak{S}^{\frac{3r}{2}}}(\mathfrak{S}^{\frac{r}{2}}x).$$

These results imply that if $\mathfrak{S}^{\frac{3r}{2}}$ enjoys Dunford's property (C), then the operators $\mathfrak{T}^q\mathfrak{S}^{\frac{r}{2}}$ and $\mathfrak{S}^{\frac{r}{2}}\mathfrak{T}^q$ inherit this property. Consequently, the local spectral behavior of higher-order operator powers controls and transfers stability to compositions involving fractional powers and iterates.

The investigation into Dunford's property (C) within this framework has thus offered new insights into the behavior of operators in Banach spaces. By generalizing existing classes, this study not only consolidates known results but also provides a foundation for further exploration. In particular, these methods suggest natural extensions to other operator classes, such as the w$\mathcal{A}(s,t)$ class, where operator equation techniques may yield analogous inheritance properties.

Future research may focus on extending the spectral inclusion framework to broader operator equations, investigating stability under perturbations, and exploring potential applications in applied mathematics and related scientific contexts.

Acknowledgment

The authors would like to thank the anonymous reviewers for their valuable remarks .

Dedication

The first author dedicates this work to the memory of his late father, Joseph Wanjala Wanyama, in gratitude for his unwavering support.

References

[1] Jinta, J., George, B., Thumbakara, R. K., & George, S. P. (2024). Understanding normal and restricted normal products in soft directed graphs. *Journal of the Nigerian Society of Physical Sciences*, **6**, 2156. https://doi.org/10.46481/jnsps.2024.2156

[2] Jinta, J., Thumbakara, R. K., Mashale, J. T., George, B., & George, S. P. (2025). Homomorphic and restricted homomorphic products of soft graphs. *Journal of the Nigerian Society of Physical Sciences*, **7**, 2391. https://doi.org/10.46481/jnsps.2025.2391

[3] Shen, J. (2018). The spectral properties of $[m]$-complex symmetric operators. *Journal of Inequalities and Applications*, **2018**, 209. https://doi.org/10.1186/s13660-018-1800-1

[4] Laursen, K. B., & Neumann, M. M. (2000). *An introduction to local spectral theory*. London Mathematical Society Monographs, New Series. Clarendon Press. https://doi.org/10.1093/oso/9780198523819.002.0003

[5] Salah, M. (2012). Bishop's property, SVEP, and Dunford property. *Electronic Journal of Linear Algebra*, **23**, 523. https://doi.org/10.13001/1081-3810.1537

[6] Hadji, S., & Zguitti, H. (2021). Back to the common spectral properties of operators satisfying $A^kB^kA^k = A^{k+1}$ and $B^kA^kB^k = B^{k+1}$. *Asian-European Journal of Mathematics*, **14**, 2150177. https://doi.org/10.1142/S1793557121501771

[7] Wurichaihu, B., & Chen, A. (2022). On local spectral properties of extended Hamilton operators. *Filomat*, **36**, 3675. https://doi.org/10.2298/FIL2211675B

[8] Elvis, A., Soto, J., & Rosas, E. (2023). Study of the property (bz) using local spectral theory methods. *Arab Journal of Basic and Applied Sciences*, **30**, 665. https://doi.org/10.1080/25765299.2023.2278217

[9] Elvis, A., Lima, L., & Sanabria, J. (2024). The property (E_A) and local spectral theory. *arXiv preprint*, arXiv:2402.10170. https://doi.org/10.48550/arXiv.2402.10170

[10] Mhamed, E., & Saber, S. (2024). Maps preserving the local spectral subspace of the Jordan product of operators. *Filomat*, **38**, 4611. https://www.pmf.ni.ac.rs/filomat-content/2024/38-13/38-13-13-20108.pdf

[11] Macias, J., Jativa, A., Bareto, J., & Aponte, E. (2025). A generalization of Zariouh's property (gaz) through local spectral theory. *Matematica Espol-Fcnm Journal*, **23**, 13. https://www.revistas.espol.edu.ec/index.php/matematica/issue/archive

[12] Bermúdez, T., Martinón, A., & Müller, V. (2024). Local spectral properties of m-isometric operators. *Journal of Mathematical Analysis and Applications*, **530**, 127717. https://doi.org/10.1016/j.jmaa.2023.127717

[13] Barens, B. A. (1998). Common operator properties of the linear operators RS and SR. *Proceedings of the American Mathematical Society*, **126**, 1055. https://doi.org/10.1090/S0002-9939-98-04218-X

[14] Lin, C., Yan, Z., & Ruan, Y. (2002). Common properties of operators RS and SR and p-hyponormal operators. *Integral Equations and Operator Theory*, **43**, 313. https://doi.org/10.1007/BF01255566

[15] Benhida, C., & Zerouali, E. H. (2006). Local spectral theory of linear operators RS and SR. *Integral Equations and Operator Theory*, **54**, 1. https://doi.org/10.1007/s00020-005-1375-3

[16] Yoo, J. K. (2008). A note on linear operators RS and SR. *Far East Journal of Mathematical Sciences*, **28**, 603. https://www.pphmj.com/abstract/3152.htm

[17] Aiena, P. (2004). *Fredholm and local spectral theory, with applications to multipliers*. Springer. https://doi.org/10.1007/1-4020-2525-4

[18] Aiena, P., & Gonzalez, M. (2011). On the Dunford property (C) for bounded linear operators RS and SR. *Integral Equations and Operator Theory*, **70**, 561. https://doi.org/10.1007/s00020-011-1875-2

[19] Benhida, C., & Zerouali, E. H. (2007). Back to RS-SR spectral theory. *Banach Center Publications*, **75**, 55. https://doi.org/10.4064/bc75-0-4

[20] Schmoeger, C. (2005). On the operator equations $ABA = A^2$ and $BAB = B^2$. *Publications de l'Institut Mathématique*, **78**, 127. https://doi.org/10.2298/pim0578127s

[21] Schmoeger, C. (2006). Common spectral properties of linear operators A and B such that $ABA = A^2$ and $BAB = B^2$. *Publications de l'Institut Mathématique*, **79**, 109. https://doi.org/10.2298/PIM0693109S

[22] Zeng, Q. P., & Zhong, H. J. (2013). New results on common properties of the bounded linear operators RS and SR. *Acta Mathematica Sinica (English Series)*, **29**, 1871. https://doi.org/10.1007/s10114-013-1758-3

[23] Zeng, Q. P., & Zhong, H. J. (2014). Common properties of bounded linear operators AC and BA: Local spectral theory. *Journal of Mathematical Analysis and Applications*, **414**, 553. https://doi.org/10.1016/j.jmaa.2014.01.021

[24] Tajmouati, A., Bakkali, A. E., & Ahmed, M. B. M. (2015). Common local spectral properties of bounded linear operators AT and SA such that BSA = ATB. *International Journal of Mathematical Analysis*, **9**, 183. https://doi.org/10.12988/ijma.2015.411347

[25] Duggal, B. P. (2011). Operator equations $ABA = A^2$ and $BAB = B^2$. *Functional Analysis, Approximation and Computation*, **3**, 9. http://operator.pmf.ni.ac.rs/www/pmf/publikacije/faac/2011/3-1-2011/faac3-1-2.pdf

[26] Triolo, S. (2020). Local spectral theory for R and S satisfying $R^nSR^n = R^j$. *Axioms*, **9**, 120. https://doi.org/10.3390/axioms9040120

[27] Pietro, A., & Gonzalez, M. (2020). Local spectral theory for operators R and S satisfying $RSR = R^2$. *Extracta Mathematicae*, **9**, 37. https://doi.org/10.3390/axioms9040120

[28] Pietro, A., & Triolo, S. (2019). Some remarks on the spectral properties of Toeplitz operators. *Mediterranean Journal of Mathematics*, **16**, 135. https://doi.org/10.1007/s00009-019-1397-8