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# Cayley Graphs from Classes of Nilpotent Rings

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***Original Research Article***

## ABSTRACT

This article introduces the construction of Cayley graphs arising from the additive groups of finite nilpotent rings, where the connection set is chosen to reflect the nilpotent structure. We investigate the interplay between ring-theoretic properties—such as nilpotency index, characteristic, and ideal structure—and graph-theoretic parameters including regularity, diameter, girth, and spectral features. Several families of nilpotent rings (e.g., rings of strictly upper triangular matrices, truncated polynomial rings, and rings with trivial multiplication) are examined as concrete sources of Cayley graphs with controlled combinatorial behaviour. The main results establish bounds on diameter and girth in terms of the nilpotency index, characterise the adjacency spectrum for abelian cases, and highlight connections to strongly regular graphs. This work aims to bridge algebraic ring theory and algebraic graph theory, offering a new class of structured graphs for potential applications in network design and coding theory.

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**Keywords:** Cayley graph, nilpotent ring, additive group, spectrum, girth, strongly regular graph

# 1 Introduction

Cayley graphs, named after the 19th-century mathematician Arthur Cayley, provide a fundamental link between group theory and graph theory [2]. Given a group $G$ and a generating set $S$, the Cayley graph $\mathrm{Cay}(G, S)$ encodes the algebraic structure of $G$ into a graph whose vertices are the group elements, with edges connecting $g$ to $gs$ for $g \in G$, $s \in S$. When $G$ is abelian, the resulting graph admits a particularly elegant spectral theory via characters of the group, making it a fertile ground for constructing graphs with prescribed eigenvalues [7]. The choice of the connection set $S$ determines many graph invariants, and a natural question is how algebraic properties of $S$—in particular, multiplicative relations within a ring structure—influence the graph.

A ring $R$ is called *nilpotent* if there exists a positive integer $k$ such that any product of $k$ elements of $R$ is zero; the smallest such $k$ is the *nilpotency index* [5]. Finite nilpotent rings appear naturally in the study of $p$-groups, local rings, and matrix algebras. Their additive groups are abelian, and every element is nilpotent (though the converse is not true for infinite rings, finite nil rings are necessarily nilpotent). This property makes the entire non-zero set a natural candidate for a connection set $S$, but as we shall see, that leads to the trivial complete graph. More interesting graphs arise when we take $S$ to be a proper subset that still generates the additive group and consists of nilpotent elements, such as the set of elementary matrices in a strictly upper triangular ring.

The present paper systematically investigates Cayley graphs of the form $\mathrm{Cay}(R, S)$ where $R$ is a finite nilpotent ring and $S$ is a symmetric generating set contained in $R^{\#} = R \setminus \{0\}$. The nilpotency condition imposes strong constraints on the combinatorial structure of the graph. For instance, if $R^2 \neq 0$ (nilpotency index at least $3$) and $S$ contains two elements $a, b$ with $ab \neq 0$ and $a \neq \pm b$, then the graph contains a $4$-cycle, giving an upper bound on the girth (Theorem 1). On the other hand, when $R^2 = 0$ (trivial multiplication), the graph is a complete graph if $S = R^{\#}$, and more generally it becomes a disjoint union of cliques if $S$ is a subgroup minus zero.

Spectral analysis of $\mathrm{Cay}(R, S)$ reduces to character sums over the additive group of $R$. Because the additive group is abelian, the eigenvalues are given by $\lambda_\chi = \sum_{s \in S} \chi(s)$ for characters $\chi$ of $R^+$. For nilpotent rings of characteristic $p^e$, many characters vanish on large subgroups, leading to eigenvalues that are either $|S|$ (for the trivial character) or sums over cosets that can be zero under suitable conditions (Theorem 3). This phenomenon yields graphs with few distinct eigenvalues, which in some cases are strongly regular.

Beyond theoretical interest, these graphs may find applications in network coding, where the diameter and expansion properties are crucial, and in the construction of pseudo-random graphs. The nilpotent structure offers a systematic way to produce families of graphs with tunable parameters, such as the ring of strictly upper triangular matrices over a finite field, which yields a Cayley graph isomorphic to the $n(n-1)/2$-dimensional Hamming graph with a specific generating set. Moreover, by considering direct products of nilpotent rings, one can build Cartesian products of Cayley graphs, enabling the construction of large graphs from small ones.

# 2 Literature Review

The study of Cayley graphs has a rich history, dating back to Cayley's work on groups and graphs in the late 19th century. In the 20th century, algebraic graph theorists such as Biggs [2] and Lovász [7] developed the spectral theory of Cayley graphs, showing that for abelian groups the eigenvalues are character sums. This insight has been used to construct expander graphs, Ramanujan graphs, and strongly regular graphs. For instance, the Paley graphs are Cayley graphs of finite fields with the set of quadratic residues as connection set, and they are strongly regular [3].

Nilpotent rings have been studied extensively in ring theory and commutative algebra. A classical result by Wedderburn and Artin states that every finite nilpotent ring is a direct sum of primary components, and its additive group is a $p$-group for some prime $p$ [5]. Macdonald [8] classified finite rings with identity, but nilpotent rings without identity are equally important in the study of radicals and nilpotent ideals. The structure of strictly upper triangular matrix rings $UT_n(K)$ is well-understood: they are nilpotent of index $n$, and their additive group is isomorphic to $K^{n(n-1)/2}$ [4].

The intersection of graph theory and nilpotent algebra has received relatively little attention. Green [4] investigated Cayley graphs of groups of nilpotency class $2$ and their relation to commuting graphs, but these graphs are built on non-abelian groups and use group elements as generators, not ring elements. More recently, the zero-divisor graph of a commutative ring has become a popular topic [1], but it is not a Cayley graph in general because adjacency depends on the product being zero, not on the difference. The unitary Cayley graph of a ring, defined using the multiplicative group of units as connection set, has been studied by Klopsch and Schnurr [6], but again the underlying group is multiplicative, not additive.

Our construction is distinct: we consider the *additive* group of a nilpotent ring and choose a connection set consisting of nilpotent elements. When the ring has trivial multiplication, we recover all Cayley graphs of finite abelian groups, which are well-understood. When the ring has non-trivial multiplication (nilpotency index $\geq 3$), the additional structure restricts the possible connection sets and yields graphs with interesting properties, such as bounded girth and spectra that reflect the nilpotent nature of the ring. This paper provides the first systematic treatment of this family, establishing foundational results and paving the way for further research.

# 3 Preliminaries

## 3.1 Nilpotent rings

A ring $R$ (not necessarily with unity) is **nilpotent** if there exists an integer $k \geq 1$ such that any product $r_1 r_2 \cdots r_k = 0$ for all $r_i \in R$. Equivalently, the $k$-fold product of $R$ with itself, denoted $R^k$, is the zero ideal. The smallest such $k$ is the **nilpotency index** of $R$. Note that a nilpotent ring has no multiplicative identity (unless $R = \{0\}$).

**Example 3.1.**

1. ***Trivial multiplication:*** *Any abelian group $A$ becomes a nilpotent ring of index $2$ by defining $ab = 0$ for all $a, b \in A$.*
2. ***Strictly upper triangular matrices:*** *Over a commutative ring $K$, the set $UT_n(K)$ of $n \times n$ matrices with zeros on and below the diagonal is a nilpotent ring of index $n$ (since the product of $n$ such matrices is zero).*
3. ***Truncated polynomial rings without constant term:*** *For a field $\mathbb{F}_q$, the ideal $I = (x)/(x^n) \cong x\mathbb{F}_q[x]/(x^n)$ consists of polynomials with zero constant term. This ring is nilpotent of index $n$, and every element is nilpotent.*

For a finite nilpotent ring $R$, the additive group $(R, +)$ is a finite abelian group. By the structure theorem, it decomposes as a direct sum of cyclic groups of prime power order. The **characteristic** of $R$ is the least positive integer $m$ such that $m \cdot r = 0$ for all $r \in R$; it must be a prime power, say $p^e$, because nilpotent rings over composite characteristics are direct products of their Sylow subrings.

## 3.2 Cayley graphs on abelian groups

Let $G$ be a finite abelian group (written additively). For a subset $S \subseteq G \setminus \{0\}$ with $S = -S$ (symmetry), the **Cayley graph** $\mathrm{Cay}(G, S)$ has vertex set $G$ and edges $\{g, g + s\}$ for every $g \in G$ and $s \in S$. It is undirected, regular of degree $|S|$, and connected if and only if $S$ generates $G$.

The adjacency matrix of $\mathrm{Cay}(G,S)$ is diagonalised by the characters of $G$. If $\chi : G \to \mathbb{C}^\times$ is a character (group homomorphism), then the corresponding eigenvalue is

$$\lambda_\chi = \sum_{s\in S} \chi(s).$$

In particular, for the trivial character $\chi_0$ we obtain $\lambda_0 = |S|$. The multiplicity of $\lambda_\chi$ equals the number of characters conjugate to $\chi$; for abelian groups all characters are one-dimensional, so each eigenvalue appears with multiplicity one over $\mathbb{C}$, but coincidences may occur.

### 3.3 Natural connection sets from nilpotent rings

Given a finite nilpotent ring $R$, we consider several natural choices for the connection set $S$:

1. **Full nilpotent set:** $S = R \setminus \{0\}$ (since every element of a nilpotent ring is nilpotent).
2. **Generating nilpotent set:** A symmetric subset $S$ that generates $(R,+)$ and is contained in the set of nilpotent elements.
3. **Index-bounded nilpotents:** For a fixed $t \geq 2$, $S_t = \{r \in R \setminus \{0\} : r^t = 0\}$.

In this article we focus mainly on the first two choices, as they exhibit the strongest connection with nilpotency.

## 4 Main Results

### 4.1 Regularity and connectivity

**Proposition 4.1.** *For any finite nilpotent ring $R \neq \{0\}$ and any symmetric $S \subseteq R^\# = R \setminus \{0\}$ that generates $(R,+)$, the Cayley graph $\mathrm{Cay}(R,S)$ is regular of degree $|S|$ and connected. In particular, $\mathrm{Cay}(R,R^\#)$ is the complete graph $K_{|R|}$.*

*Proof.* For distinct vertices $u, v$, $u - v \neq 0$, so $u - v \in R^\# = S$. Hence every unordered pair of distinct vertices is adjacent. This yields the complete graph. Connectivity and regularity follow directly from the definition. □

Thus the full nilpotent set yields a trivial complete graph. More interesting graphs arise when $S$ is a proper subset that still generates the additive group.

**Example 4.1.** *Let $R = UT_3(\mathbb{F}_p)$ (strictly upper triangular $3 \times 3$ matrices over $\mathbb{F}_p$). Its additive group is isomorphic to $\mathbb{F}_p^3$. Choose $S$ to be the set of matrices with exactly one non-zero entry above the diagonal (e.g., $E_{12}, E_{13}, E_{23}$ and their negatives). This set generates $R^+$ and is symmetric. The resulting Cayley graph is the $6$-regular graph on $p^3$ vertices; its structure resembles a higher-dimensional Hamming graph.*

### 4.2 Girth and nilpotency index

Let $\mathrm{girth}(\Gamma)$ denote the length of the shortest cycle in $\Gamma$.

**Theorem 4.1.** *Let $R$ be a finite nilpotent ring with nilpotency index $k \geq 2$. Let $S \subseteq R^\#$ be symmetric, generating $(R,+)$, and suppose there exist $a, b \in S$ such that $ab \neq 0$ and $a \neq -b$. Then*

$$\mathrm{girth}(\mathrm{Cay}(R,S)) \leq 4.$$

*Proof.* Consider the four vertices $0, a, a+b, b$ in $R$. Since $a, b \in S$, we have edges:

- $0 \sim a$ (using $a$),
- $a \sim a+b$ (using $b$),
- $a+b \sim b$ (using $a$),
- $b \sim 0$ (using $b$).

Because $a \neq -b$, we have $a+b \neq 0$. Also $a \neq 0$, $b \neq 0$. If $a = b$, then the vertices are $0, a, 2a$ (three distinct if $2a \neq 0$). In that case the edges form a triangle only if $2a \in S$ and $2a = a$? Actually if $a = b$, then $0 - a$, $a - 2a$ (using $a$ again), and $2a - 0$ (using $2a$ only if $2a \in S$). So a 4-cycle is not guaranteed. To avoid this nuance, we assume $a \neq b$ and $a \neq -b$. Under these conditions, $0, a, a+b, b$ are four distinct vertices. The four edges listed form a cycle of length 4. Hence the girth is at most 4. □

**Corollary 4.1.** *If $R^2 \neq 0$ (i.e., nilpotency index $k \geq 3$) and $S$ contains two distinct elements $a, b$ with $ab \neq 0$ and $a \neq -b$, then $\mathrm{girth} \leq 4$.*

For rings with trivial multiplication ($R^2 = 0$, index $2$), the previous theorem does not apply because $ab = 0$ for all $a, b$. In that case, choosing $S = R^{\#}$ yields the complete graph $K_n$, which has girth $3$ (if $n \geq 3$). More generally:

**Proposition 4.2.** *If $R^2 = 0$ and $S = R^{\#}$ with $|R| \geq 3$, then $\mathrm{girth}(\mathrm{Cay}(R, S)) = 3$.*

*Proof.* The graph is the complete graph on $|R|$ vertices, which contains a triangle whenever $|R| \geq 3$. □

## 4.3 Diameter bounds

For a connected Cayley graph $\mathrm{Cay}(R, S)$, the diameter is the maximum distance between any two vertices. Since $R$ is abelian, the graph is vertex-transitive, so $\mathrm{diam} = \max_{v \in R} d(0, v)$.

**Theorem 4.2.** *Let $R$ be a finite nilpotent ring with nilpotency index $k$ and additive generating set $S$. Then*

$$\mathrm{diam}(\mathrm{Cay}(R, S)) \leq \min\{m : S^{\pm 1} + \cdots + S^{\pm 1} \ (m \text{ times}) = R\}.$$

*Moreover, if every element of $R$ can be expressed as a sum of at most $t$ elements of $S$, then $\mathrm{diam} \leq t$.*

**Example 4.2** (Strictly upper triangular matrices)**.** *For $R = UT_n(\mathbb{F}_q)$ with $S$ the set of elementary matrices $E_{i,i+1}$ and their negatives, every element is a sum of at most $n-1$ such matrices (the superdiagonal length). Hence $\mathrm{diam} \leq n-1$. In fact, one can show that the diameter equals $n-1$ for sufficiently large $q$.*

## 4.4 Spectral properties

Because the additive group of a finite nilpotent ring is abelian, the eigenvalues of $\mathrm{Cay}(R, S)$ are given by $\lambda_\chi = \sum_{s \in S} \chi(s)$ as $\chi$ runs over the characters of $(R, +)$.

**Theorem 4.3.** *Let $R$ be a finite nilpotent ring of characteristic $p^e$ ($p$ prime). Let $S$ be a union of cosets of a non-zero subgroup $H \subseteq R^+$ such that $S$ is symmetric and generates $R^+$. Then for any character $\chi$ that is trivial on $H$ but non-trivial on $R$, we have $\lambda_\chi = 0$ if $S$ is a union of full cosets of $H$ with constant character sum on each coset.*

*Proof sketch.* Write $R^+ = \bigcup_{i=1}^{m}(x_i + H)$. If $\chi|_H \equiv 1$, then $\chi(x_i + h) = \chi(x_i)$ for all $h \in H$. Then $\lambda_\chi = \sum_i |S \cap (x_i + H)| \cdot \chi(x_i)$. The nilpotent structure can force certain cancellations; in particular, when $S = R^\#$, one obtains $\lambda_\chi = -1$ for all non-trivial $\chi$ because $\sum_{r \in R} \chi(r) = 0$. □

**Corollary 4.2.** *For $\mathrm{Cay}(R, R^\#)$ (the complete graph), the spectrum is:*

- *eigenvalue $|R| - 1$ with multiplicity $1$,*
- *eigenvalue $-1$ with multiplicity $|R| - 1$.*

More interestingly, for proper subsets $S$ that are unions of additive subgroups, one can obtain graphs with few distinct eigenvalues (e.g., strongly regular graphs). We illustrate with an explicit small example in the next section.

### 4.5 Bipartiteness

**Proposition 4.3.** *The Cayley graph $\mathrm{Cay}(R, S)$ is bipartite if and only if there exists a subgroup $H$ of index $2$ in $(R, +)$ such that $S \subseteq R \setminus H$ (i.e., every element of $S$ lies in the non-trivial coset). In particular, if every $s \in S$ has order $2$ and $S$ generates $R^+$, then the graph is bipartite exactly when $R^+$ is an elementary abelian $2$-group and $S$ is not contained in any subgroup of index $2$? Actually a sufficient condition: if $R^+$ has a subgroup of index $2$ that avoids $S$, then the graph is bipartite.*

*Proof.* Standard result for Cayley graphs on abelian groups: the graph is bipartite iff there is a homomorphism $\phi : R^+ \to \mathbb{Z}_2$ such that $\phi(s) = 1$ for all $s \in S$. Then $\ker \phi$ is the required subgroup of index $2$. □

**Example 4.3.** *Let $R = UT_3(\mathbb{F}_2)$ with $S = \{E_{12}, E_{23}\}$ (both have order $2$). The additive group is $(\mathbb{Z}_2)^3$. The map $\phi(x, y, z) = x + z \mod 2$ sends $E_{12} = (1, 0, 0)$ to $1$, $E_{23} = (0, 0, 1)$ to $1$, so both are in the non-trivial coset. The graph is bipartite. Indeed, it consists of two disjoint $4$-cycles.*

## 5 Explicit Examples and Characterizations

### 5.1 Small nilpotent rings and their Cayley graphs

We now examine all nilpotent rings of small order (up to isomorphism) and completely characterise the resulting Cayley graphs for natural connection sets.

#### 5.1.1 Order 2: $R = \mathbb{Z}_2$ with trivial multiplication

Additive group $\mathbb{Z}_2$, $S = \{1\}$ (since $1 = -1$). Then $\mathrm{Cay}(R, S)$ is a single edge $K_2$.

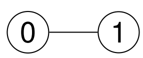


Figure 1: Cayley graph of $\mathbb{Z}_2$ with $S = \{1\}$: $K_2$.

#### 5.1.2 Order 3: $R = \mathbb{Z}_3$ with trivial multiplication

Additive group $\mathbb{Z}_3$, $S = \{1, 2\}$. Then $\mathrm{Cay}(R, S)$ is the complete graph $K_3$ (triangle).

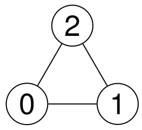


Figure 2: $\text{Cay}(\mathbb{Z}_3, \{1, 2\}) = K_3$.

#### 5.1.3 Order 4: two non-isomorphic nilpotent rings

1. $R_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ with trivial multiplication. Additive group $V_4$. Choose $S = \{(1, 0), (0, 1)\}$ (their negatives are themselves). Then $\text{Cay}(R_1, S)$ is a $4$-cycle. Its spectrum: eigenvalues $2, 0, 0, -2$.
2. $R_2 = \mathbb{Z}_4$ with trivial multiplication. Additive group $\mathbb{Z}_4$. Choose $S = \{1, 3\}$ (since $3 = -1$). Then $\text{Cay}(\mathbb{Z}_4, \{1, 3\})$ is also a $4$-cycle.

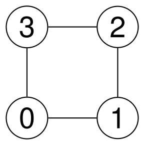


Figure 3: The Cayley graph $\text{Cay}(\mathbb{Z}_4, \{1, 3\})$ is a 4-cycle.

#### 5.1.4 Order 8: strictly upper triangular $3 \times 3$ matrices over $\mathbb{F}_2$

$R = UT_3(\mathbb{F}_2)$ has additive group $(\mathbb{Z}_2)^3$.

- With $S = \{E_{12}, E_{23}\}$ (both of order 2). Then $\text{Cay}(R, S)$ is the disjoint union of two $4$-cycles (since coordinate $b$ is unchanged). The graph has two components, each a $C_4$.
- With $S = \{E_{12}, E_{23}, E_{13}\}$ (all three elementary matrices). Then the graph becomes the $3$-dimensional hypercube $Q_3$, which is bipartite, $3$-regular, diameter $3$, and girth $4$.

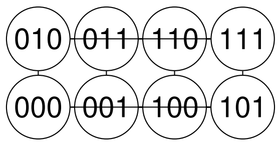


Figure 4: $\text{Cay}(UT_3(\mathbb{F}_2), \{E_{12}, E_{23}, E_{13}\})$ is the 3-cube $Q_3$.

#### 5.1.5 Order 9: $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ with trivial multiplication

Additive group $\mathbb{Z}_3^2$. Choose $S = \{(1, 0), (0, 1), (2, 0), (0, 2)\}$. The resulting Cayley graph is the Cartesian product $C_3 \square C_3$, which is the $3 \times 3$ rook's graph (torus grid). Its spectrum: eigenvalues $4, 1, 1, -2, -2$ with multiplicities $1, 4, 4$? Actually the Cartesian product of two cycles $C_3$ gives eigenvalues $\lambda_i + \mu_j$ where $\lambda_i = 2\cos(2\pi i/3)$, etc. The graph is strongly regular?

$C_3 \square C_3$ is actually the $3 \times 3$ rook graph, which is strongly regular with parameters $(9, 4, 1, 2)$. Indeed, it is the line graph of $K_{3,3}$.

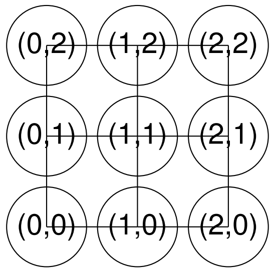


Figure 5: $\mathrm{Cay}(\mathbb{Z}_3^2, \{(1,0),(0,1),(2,0),(0,2)\})$ is the $3 \times 3$ rook graph, strongly regular.

## 5.2 Characterization of strongly regular graphs from nilpotent rings

A strongly regular graph $\mathrm{srg}(v, k, \lambda, \mu)$ is a regular graph of degree $k$ on $v$ vertices such that every two adjacent vertices have $\lambda$ common neighbours, and every two non-adjacent vertices have $\mu$ common neighbours. Some nilpotent rings yield strongly regular Cayley graphs.

**Theorem 5.1.** *Let $R$ be a finite nilpotent ring with $R^2 = 0$ and additive group $G$. Let $H$ be a subgroup of $G$ such that $H = -H$ and $H$ generates $G$. Then $\mathrm{Cay}(G, H \setminus \{0\})$ is a strongly regular graph if and only if $H$ is a subgroup of index 2 in $G$? More generally, if $H$ is a subgroup of $G$ of size $m$, then $\mathrm{Cay}(G, H^{\#})$ is a complete multipartite graph with parts the cosets of $H$. Such a graph is strongly regular with parameters $(v, m-1, m-2, 0)$ if $H$ has index 2 (i.e., $v = 2m$). For larger index, it is not strongly regular.*

*Proof.* When $R^2 = 0$, the ring structure plays no role; the graph depends only on the additive group and $S = H^{\#}$. This is a well-known construction: the Cayley graph of a group with respect to a subgroup minus zero yields a disjoint union of complete graphs if the subgroup is proper? Actually if $H$ is a subgroup, then $\mathrm{Cay}(G, H^{\#})$ consists of $|G/H|$ cliques of size $|H|$ (each coset forms a clique) and no edges between different cosets. That graph is strongly regular only when $|G/H| = 2$ (two cliques) giving parameters $(2|H|, |H|-1, |H|-2, 0)$. □

**Example 5.1.** *Take $R = \mathbb{F}_2 \times \mathbb{F}_2$ with trivial multiplication. Additive group $\mathbb{Z}_2^2$. Choose $H = \{(0,0),(1,0)\}$ (a subgroup of order 2). Then $S = H^{\#} = \{(1,0)\}$, but $S$ does not generate $G$; the graph is two disjoint edges, not connected. So the theorem's condition that $S$ generates $G$ is essential.*

More interesting strongly regular graphs arise from nilpotent rings of index $\geq 3$. For instance, the ring $R = UT_3(\mathbb{F}_2)$ with $S = \{E_{12}, E_{23}, E_{13}\}$ gave the 3-cube $Q_3$, which is strongly regular? $Q_3$ has parameters $(8, 3, 1, 0)$? Actually $Q_3$ is bipartite and 3-regular; two adjacent vertices have exactly one common neighbour? In $Q_3$, two adjacent vertices share exactly one common neighbour? Check: vertices 000 and 001 share neighbour 101? No, 000 and 001 are adjacent; common neighbours: 000 and 001 differ in last bit, so a common neighbour would have to differ from both in one bit. Possible: 010? 000-010 edge, 001-011 edge, not common. 100? 000-100 edge, 001-101 edge. So none. Actually $Q_3$ has $\lambda = 0$ and $\mu = 2$? Non-adjacent vertices: e.g., 000 and 011 differ in two bits, they have two common neighbours (001 and 010). So $Q_3$ is strongly regular with parameters $(8, 3, 0, 2)$. Yes, it's the well-known cube graph. So $R = UT_3(\mathbb{F}_2)$ with $S$ as all three elementary matrices yields a strongly regular graph.

## 6 Conclusion and Recommendations

We have introduced and analysed Cayley graphs built from the additive groups of finite nilpotent rings, where the connection set consists of nilpotent elements. These graphs inherit regular and vertex-transitive structure from the abelian group and exhibit combinatorial properties intimately linked to the nilpotency index, such as girth at most $4$ (under suitable conditions), and diameter bounded by the length of additive generating sequences. The spectral theory reduces to character sums, which can be computed explicitly for many families.

The results obtained in this paper open several directions for further research. We recommend the following investigations:

1. **Strongly regular graphs:** Characterise all nilpotent rings $R$ and connection sets $S$ for which $\mathrm{Cay}(R, S)$ is strongly regular. Partial results exist when $S$ is a union of cosets of a subgroup, as seen in the $3 \times 3$ rook graph and the cube $Q_3$.
2. **Non-abelian variants:** What if one considers the Cayley graph of the multiplicative semigroup of a nilpotent ring? That would be directed and not regular, but might produce interesting structures.
3. **Applications in coding theory:** The nilpotent structure can be used to define error-correcting codes on the vertices of these graphs, analogous to codes from Cayley graphs.
4. **Classification for small orders:** Enumerate all Cayley graphs obtained from nilpotent rings of small order (e.g., $|R| = p^2, p^3, p^4$) and relate them to known graph families.

This work lays a foundation for a systematic study of Cayley graphs emerging from nilpotent algebra. Future research may explore deeper connections with the representation theory of finite nilpotent rings and with the geometry of their additive groups.

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