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A Probabilistic Approach to Solving Unknown Nonlinear Stochastic Differential Equations with Bayesian Uncertainty Quantification and Inference

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Abstract

This study develops a unified Bayesian framework for solving nonlinear stochastic differential equations (SDEs) with unknown dynamics, treating their solution as a problem of probabilistic inference rather than deterministic computation. Classical numerical solvers typically assume known system dynamics and ignore modeling or discretization errors, leading to overconfident predictions. To address these limitations, the proposed approach simultaneously reconstructs hidden state trajectories, estimates unknown drift and diffusion terms, and rigorously quantifies uncertainty. The method integrates the Fokker–Planck and Kushner–Stratonovich equations within a continuous-time Bayesian filtering framework, enabling full propagation of state distributions and recursive assimilation of measurements. Analytical derivations for the mean and covariance provide practical tools for uncertainty quantification, while the probabilistic formulation ensures consistent joint estimation of states and parameters. Results show that embedding uncertainty directly into the solution process enhances reliability, facilitates data-driven model discovery, and provides a principled foundation for sensitivity analysis and risk-aware decision-making in complex stochastic systems.

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1 Introduction

Nonlinear stochastic differential equations (SDEs) provide a unifying framework for modeling complex dynamical systems under intrinsic randomness, with applications across physics, biology, finance, and engineering. However, classical analytical and numerical solvers often fail when the underlying dynamics are unknown, strongly nonlinear, or lack closed-form solutions. Moreover, conventional solvers typically neglect uncertainty quantification, producing overconfident and potentially unreliable predictions [5].

Building on the emerging paradigm of probabilistic numerics, numerical computation can be reframed as a problem of statistical inference, yielding posterior distributions over solutions rather than single point estimates. This perspective has been successfully developed for ordinary differential equations through Gaussian process priors and Bayesian filtering methods [1, 16], as well as through probabilistic integrators that interpret time discretization schemes as stochastic processes to quantify numerical error. Despite this progress, nonlinear SDEs with unknown dynamics remain underexplored, where both the latent state trajectories and governing functions must be inferred simultaneously.

To address these challenges, a Bayesian formulation offers a principled solution. By unifying the Fokker–Planck and Kushner–Stratonovich equations with probabilistic inference, it becomes possible to propagate full state distributions, assimilate noisy observations, and jointly recover unknown drift and diffusion terms in a single coherent framework [4]. This synthesis reflects the broader vision of probabilistic numerics, where computation itself is treated as a source of quantifiable uncertainty rather than an opaque numerical artifact [6, 2], thereby motivating rigorous Bayesian methods to characterize nonlinear stochastic dynamics with principled uncertainty quantification.

2 Problem Statement

Classical solvers and filters for nonlinear stochastic differential equations (SDEs) assume fully specified dynamics and negligible modeling error, assumptions rarely satisfied in practice. In real systems, drift, control, and diffusion terms are often unknown, time-varying, and observed only through noisy, incomplete data. Existing methods cannot jointly reconstruct hidden states, identify governing dynamics, and rigorously quantify uncertainty within a coherent Bayesian framework. This gap undermines reliable modeling, inference, and control of complex stochastic systems, highlighting the need for a unified approach that integrates state estimation, model discovery, and uncertainty quantification in a principled manner.

3 Methods

3.1 Model Formulation

The nonlinear system dynamics were modeled using an Itô stochastic differential equation (SDE) with additive process and measurement noise.

$$Dynamic \quad model: df_t = H(f_t, t)dt + U(f_t)dt + G(f_t, t)d\eta_t; \quad t_0 \leq t \tag{1}$$

$$Measurement \quad model: dy_t = S(f_t, t)dt + Q^{1/2}(t)d\beta_t \tag{2}$$

Where f_t denotes the state vector with $H(f_t, t)$ and $G(f_t, t)$ representing the nonlinear drift and diffusion functions, respectively and $U(f_t)$ an additional bias term introduced to account for model discrepancies or discretization errors. Process noise $\{d\eta_t, t > t_0\}$ is modeled as Brownian motion. The output y_t is related to the state by the nonlinear measurement model $S(f_t, t)$ with measurement noise covariance $Q(t)$ and additive Brownian motion noise $\{d\beta_t, t > t_0\}$. To capture uncertainty in state evolution, the system is described probabilistically by a time-varying density function rather than as a deterministic trajectory. This density satisfies the Fokker–Planck equation (FPE), derived from the Chapman–Kolmogorov equation using a small-increment expansion [13]. Application of Itô calculus and a Taylor series expansion isolates the drift and diffusion terms, yielding a partial differential equation that conserves probability and governs the evolution of the state density.

3.2 Inference via Bayesian Filtering

To incorporate measurement updates, the Kushner–Stratonovich equation (KSE) was formulated within a continuous-time Bayesian filtering framework [7]. The measurement likelihood was linearized for small time steps, producing an incremental correction term. This correction was then combined with the FPE prediction to obtain the full probabilistic equation, allowing recursive state estimation conditioned on available observations.

3.3 Moments Computation

The time evolution of the state mean was obtained by differentiating the expected value of the state and substituting the FPE into the resulting expression. Integration by parts was employed to eliminate boundary contributions under the assumption that the density vanishes at infinity. The covariance dynamics were derived from its definition using Itô's Lemma [11, 12], which accounts for both deterministic and stochastic contributions to uncertainty propagation. A first-order Taylor expansion (Jacobian linearization) of the nonlinear drift term was used to obtain a tractable expression for the covariance evolution, explicitly separating the contributions of drift and diffusion.

This analytical framework adopts a fully probabilistic and Bayesian approach to characterize uncertainty. While the full state distribution evolves according to the FPE–KSE formulation, low-order statistical moments (mean and covariance) were propagated for practical uncertainty quantification. Key assumptions include Gaussian Brownian motion increments and vanishing boundary conditions, ensuring the validity of integrations and moment derivations.

4 Results and Discussion

In this section, we develop a probability-based framework to estimate, approximate, or infer solutions to nonlinear SDEs. This approach employs probabilistic tools specifically Bayesian inference to model the behavior of the unknown solution. Rather than representing the solution as a deterministic function, it is modeled as a probability distribution. This framework provides probabilistic estimates of the solution at various points in time, explicitly accounting for uncertainty and noise in the system.

4.1 Derivation of The Fokker-Planck Kolmogorov Equation

The Fokker−Planck Kolmogorov Equation describes the evolution of the probability density function $p(f_t, t)$ for a stochastic process governed by a Stochastic Differential Equation (SDE) [13]. Here we derive the FPE and the Kushner−Stratonovich Equation for the given nonlinear dynamic and measurement models. The mean and covariance expressions are also derived, these equations are fundamental for applying particle filters to non-linear and non-Gaussian systems.

$$Dynamic \quad model : df_t = H(f_t, t)dt + U(f_t)dt + G(f_t, t)d\eta_t; \quad t_0 \le t \tag{3}$$

$$Measurement \quad model : dy_t = S(f_t, t)dt + Q^{1/2}(t)d\beta_t \tag{4}$$

We now derive the FPE and the KSE [9, 10, 15] for these models and compute the mean and covariance. We begin by deriving the FPE for $p(f_t, t)$, the state probability density . The FPE describes the time evolution of the unconditional probability density $p(f_t, t)$ for the dynamic model. The probability density $p(f_t, t)$ evolves according to the Chapman−Kolmogorov equation.

$$p(f_t, t + \Delta t) = \int p(f_t \mid f_{t-\Delta t})p(f_{t-\Delta t}, t)df_{t-\Delta t} \tag{5}$$

here $p(f_t \mid f_{t-\Delta t})$ is the transition probability, $p(f_{t-\Delta t}, t)$ is the density at the earlier time, The transition probability $p(f_t \mid f_{t-\Delta t})$ depends on the dynamics described by the SDE

$$df_t = H(f_t, t)dt + U(f_t)dt + G(f_t, t)d\eta_t \tag{6}$$

expanding the SDE for small time increment Δt, we approximate

$$f_t \approx f_{t-\Delta t} + \Big[H(f_{t-\Delta t}, t) + U(f_{t-\Delta t})\Big]\Delta t + G\Big(f_{t-\Delta t}, t\Big)\Delta\eta_t \tag{7}$$

where $\Delta\eta_t \sim \mathcal{N}(0, \Delta t)$ is a Gaussian random variable. Let $p(f_t, t)$ be the probability density function of f_t. Expanding $p(f_t, t + \Delta t)$ using a Taylor's series gives

$$p(f_t, t + \Delta t) = p(f_t, t) + \Delta t\frac{\partial p(f_t, t)}{\partial t} + O(\Delta t^2) \tag{8}$$

For the transition density $p(f_t \mid f_{t-\Delta t})$, we expand

$$p\Big(f_t \mid f_{t-\Delta t}\Big) = \delta\Big(f_t - \Big[f_{t-\Delta t} + (H + U)\Delta t + G\Delta\eta_t\Big]\Big) \tag{9}$$

here $\delta(.)$ is the Dirac delta function. The mean and variance of $\Delta\eta_t$ contribute to the drift and diffusion terms. The drift term $H(f_t, t) + U(f_t)$ leads to the first−order derivative, while the diffusion term $G(f_t, t)$ contributes to the second−order derivatives.

$$\mathbb{E}[\Delta f_t] = [H(f_t, t) + U(f_t)]\Delta t \tag{10}$$

The expectation of the drift term over small increments Δt leads to

$$\frac{\partial}{\partial f_t}\Big([H(f_t, t) + U(f_t)]p(f_t, t)\Big) \tag{11}$$

The variance of the diffusion term leads to the second−order derivative. The variance of Δf_t is

$$\mathbb{E}[(\Delta f_t)^2] = G^2(f_t, t)\Delta t \tag{12}$$

This contributes to

$$\frac{1}{2}\frac{\partial^2}{\partial f_t^2}\Big(G^2(f_t, t)p(f_t, t)\Big) \tag{13}$$

Combining the drift and the diffusion terms, the evolution of the density $p(f_t, t)$ is governed by

$$\frac{\partial}{\partial t}p(f_t, t) = -\frac{\partial}{\partial f_t}\Big([H(f_t, t) + U(f_t)]p(f_t, t)\Big) + \frac{1}{2}\frac{\partial^2}{\partial f_t^2}\Big(G^2(f_t, t)p(f_t, t)\Big) \tag{14}$$

Which is the required FPE. The drift term models the deterministic flow of probability mass and the diffusion term accounts for the spreading of the density due to stochastic fluctuations. This FPE applies to the general nonlinear drift $H(f_t, t) + U(f_t)$ and diffusion $G(f_t, t)$.

4.2 Derivation of The Kushner−Stratonovich Equation

The Kushner−Stratonovich Equation (KSE) describes the evolution of the conditional probability density $p(f_t, t))$ given measurements $y_{[0,t]}$ combines the state dynamics via the FPE and the measurements updates via Baye's rule [7]. The KSE is

$$\frac{\partial p(f_t \mid y_{[0,t]})}{\partial t} = -\frac{\partial}{\partial f_t}\Big(H(f_t, t) + U(f_t)p(f_t \mid y_{[0,t]}\Big) + \frac{1}{2}\frac{\partial^2}{\partial f_t^2}\Big(G^2(f_t, t)p(f_t \mid y_{[0,t]}\Big) + Correction \tag{15}$$

To derive the Bayesian correction term also known as the measurement correction term in the KSE, we begin with the principle of Bayesian filtering and how measurements updates the conditional density. The goal is to find the conditional density $p(f_t \mid y_{[0,t]})$, given the prior density of $p(f_t)$ derived from the dynamic model (1) and it's evolution governed by the FPE (12). The likelihood of observations dy_t is derived from the measurement model (2). By Baye's rule

$$p(f_t \mid y_{[0,t]}) \propto p(y_t \mid f_t)p(f_t) \tag{16}$$

where $p(y_t \mid f_t)$ is the likelihood of the measurement given the state. The measurement model (2) implies that dy_t is conditionally Gaussian with mean $\mathbb{E}[dy_t \mid f_t] = S(f_t, t)dt$ and covariance $cov(dy_t \mid f_t) = Q(t)dt$. The likelihood $p(y_t \mid f_t)$ can be expressed as

$$p\Big(y_t \mid f_t\Big) \propto exp\Big(-\frac{1}{2}(dy_t - S(f_t, t))dt\Big)^T Q^{-1}(t)\Big(dy_t - S(f_t, t)dt\Big) \tag{17}$$

the conditional density $p(f_t \mid y_{[0,t]})$ evolves according to

$$\frac{\partial}{\partial t}p(f_t \mid y_{[0,t]}) = Fokker - Planckdynamics + Measurementupdate \tag{18}$$

The measurement update term arises from the likelihood $p(y_t \mid f_t)$ that is

$$p(f_t \mid y_{[0,t]}) \propto p(y_t \mid f_t)p(f_t) \tag{19}$$

Expanding this, we account for the new observation dy_t incrementally, which gives us

$$p\Big(f_t \mid y_{[0,t+dt]}\Big) = p\Big(f_t \mid y_{[0,t]}\Big)exp\Big(-\frac{1}{2}(dy_t - S(f_t,t)dt)^T Q^{-1}(t)(dy_t - S(f_t,t)dt)\Big) \tag{20}$$

Taking logarithms and linearizing for small dt, we have

$$Logp\Big(f_t \mid y_{[0,t+dt]}\Big) = Logp\Big(f_t \mid y_{[0,t]}\Big) - \frac{1}{2}\Big(dy_t - S(f_t,t)dt\Big)^T Q^{-1}(t)\Big(dy_t - S(f_t,t)dt\Big) \tag{21}$$

Differentiating with respect to t, we have

$$\frac{\partial}{\partial t}\log p\big(f_t \mid y_{[0,t]}\big) = \big(dy_t - S(f_t,t)\,dt\big)^\top Q^{-1}(t)\,\big(S(f_t,t) - \mathbb{E}[\,S(f_t,t)\,]\big). \tag{22}$$

Expanding the expectation term, we obtain

$$dy_t - \mathbb{E}[\,dy_t\,] \;=\; dy_t - \mathbb{E}[\,S(f_t,t)\,]\,dt. \tag{23}$$

The term $\big(S(f_t,t) - \mathbb{E}[\,S(f_t,t)\,]\big)$ adjusts the conditional density $p\big(f_t \mid y_{[0,t]}\big)$ based on how the state affects the measurement. Combining these expressions, the correction term becomes

$$p\big(f_t \mid y_{[0,t]}\big)\,\big(S(f_t,t) - \mathbb{E}[\,S(f_t,t)\,]\big)^\top Q^{-1}(t)\,\big(dy_t - \mathbb{E}[\,dy_t\,]\big). \tag{24}$$

Therefore, the Kushner–Stratonovich equation can be written as

$$\begin{aligned}\frac{\partial}{\partial t}p(f_t,t) = -\frac{\partial}{\partial f_t}\Big([\,H(f_t,t) + U(f_t)\,]\,p(f_t,t)\Big) \;+\; \frac{1}{2}\,\frac{\partial^2}{\partial f_t^2}\Big(G^2(f_t,t)\,p(f_t,t)\Big) \\ +\; p\big(f_t \mid y_{[0,t]}\big)\,\big(S(f_t,t) - \mathbb{E}[\,S(f_t,t)\,]\big)^\top Q^{-1}(t)\,\big(dy_t - \mathbb{E}[\,dy_t\,]\big).\end{aligned} \tag{25}$$

The correction term updates $p\big(f_t \mid y_{[0,t]}\big)$ to account for non-Gaussian, nonlinear effects in the measurements.

4.3 Moments of The Probabilistic Model

In this section, we focus on deriving the mean and covariance of the state variables, which are essential descriptors in the analysis of stochastic systems. The mean provides a measure of the average or expected behavior of the system over time. It helps us understand where the system is likely to be centered, offering insights into its typical trajectory under uncertainty. The covariance, on the other hand, characterizes how different components of the state vary together. It not only reflects the spread or dispersion of the state values but also captures how fluctuations in one component might be related to changes in another. In uncertainty quantification, the mean and covariance are used to assess the reliability of simulations and to gauge the influence of uncertain parameters or external inputs [3].

4.3.1 Derivation of the Mean

The mean of the state f_t is given by

$$\mathbb{E}[f_t] = \int f_t p(f_t, t) df_t \tag{26}$$

where $p(f_t, t)$ is the probability density function of f_t at time t. $\mathbb{E}[f_t]$ represents the expected value (mean) of the state f_t. We want to derive the time evolution of the mean $\frac{d}{dt}\mathbb{E}[f_t]$ using the FPE [14]. Differentiating $\mathbb{E}[f_t]$ with respect to time t, we have.

$$\frac{d}{dt}\mathbb{E}[f_t] = \frac{d}{dt}\int f_t p(f_t, t) df_t \tag{27}$$

Since f_t is independent of time, the time derivative only acts on $p(f_t, t)$, which gives us

$$\frac{d}{dt}\mathbb{E}[f_t] = \int f_t \frac{\partial p(f_t, t)}{\partial t} f_t \tag{28}$$

Substituting the FPE into the time derivative of the mean, we have

$$\frac{d}{dt}\mathbb{E}[f_t] = \int f_t \Big[-\frac{\partial}{\partial f_t}\Big([H(f_t, t) + U(f_t, t)]p(f_t, t)\Big) + \frac{1}{2}\frac{\partial^2}{\partial f_t^2}\Big(G^2(f_t, t)p(f_t, t)\Big)\Big] df_t \tag{29}$$

The first term involves a first-order derivative.

$$\int f_t \Big[-\frac{\partial}{\partial f_t}\Big([H(f_t, t) + U(f_t, t)]p(f_t, t)\Big)\Big] df_t \tag{30}$$

Using integration by parts, where $u = f_t$, $dv = \frac{\partial}{\partial f_t}\Big([H(f_t, t) + U(f_t)]p(f_t, t)\Big) df_t$ and assuming the boundary terms vanish (probability density decays to zero as $f_t \rightarrow \pm\infty$), we get

$$\int f_t \Big[-\frac{\partial}{\partial f_t}\Big([H(f_t, t) + U(f_t, t)]p(f_t, t)\Big)\Big] df_t = \int H\Big[(f_t, t) + U(f_t)\Big] p\Big(f_t, t\Big) df_t \tag{31}$$

The second term involves a second-order derivative

$$\int f_t \frac{1}{2}\frac{\partial^2}{\partial f_t^2}\Big(G^2(f_t, t)p(f_t, t)\Big) df_t \tag{32}$$

Using integration by parts twice and assuming the boundary terms vanish, the contribution from this term simplifies to zero

$$\int f_t \frac{1}{2}\frac{\partial^2}{\partial f_t^2}\Big(G^2(f_t, t)p(f_t, t)\Big) df_t = 0 \tag{33}$$

Combining these results, we have

$$\frac{d}{dt}\mathbb{E}[f_t] = \int \Big[H(f_t,t) + U(f_t)\Big]p\Big(f_t,t\Big)df_t \tag{34}$$

Expressing the results in terms of expectation, we get

$$\frac{d}{dt}\mathbb{E}[f_t] = \mathbb{E}[H(f_t,t) + U(f_t)] \tag{35}$$

Which is the time evolution of the mean $\mathbb{E}[f_t]$. The mean evolves according to the expected value of the drift term $H(f_t,t)$ and the control term $U(f_t)$. The diffusion term $G(f_t,t)$ does not directly affect the mean but contributes to the evolution of the covariance.

4.3.2 Derivation of the Covariance

The covariance matrix $\sum_t$ of the state f_t is defined as

$$\sum_t = \mathbb{E}\Big[f_t - \mathbb{E}[f_t](f_t - \mathbb{E}[f_t])^T\Big] \tag{36}$$

Our goal is to derive the time evolution of $\sum_t$, denoted by $\frac{d\sum_t}{dt}$ using the stochastic dynamics and properties of the underlying processes [8]. Taking the time derivative of $\sum_t$, we have

$$\frac{d\sum_t}{dt} = \frac{d}{dt}\mathbb{E}\Big[(f_t - \mathbb{E}[f_t])(f_t - \mathbb{E}(f_t))^T\Big] \tag{37}$$

Using the product rule and the linearity of expectations, we expand and obtain

$$\frac{d\sum_t}{dt} = \mathbb{E}\Big[\frac{d}{dt}(f_t - \mathbb{E}[f_t])(f_t - \mathbb{E}[f_t])^T\Big] + \mathbb{E}\Big[(f_t - \mathbb{E}[f_t])\frac{d}{dt}(f_t - \mathbb{E}[f_t])^T\Big] \tag{38}$$

Since $\frac{d}{dt}\mathbb{E}[f_t] = \mathbb{E}[H(f_t,t) + U(f_t)]$, the time evolution of $\mathbb{E}[f_t]$ only appears indirectly through f_t. The dynamics of f_t are governed by (1). The infinitesimal change in f_t is thus

$$\frac{df_t}{dt} = H(f_t,t) + U(f_t) + G(f_t,t)\frac{d\eta_t}{dt} \tag{39}$$

Using the dynamics and $It\hat{o}'s$ Lemma, the covariance evolution can be expressed as

$$\begin{aligned}\frac{d\sum_t}{dt} = \mathbb{E}\Big[(H(f_t,t) + U(f_t))(f_t - \mathbb{E}[f_t])^T\Big] + \mathbb{E}\Big[(f_t - \mathbb{E}[f_t])(H(f_t,t) + U(f_t))^T\Big] \\ + \mathbb{E}\Big[G(f_t,t)G(f_t,t)^T\Big]\end{aligned} \tag{40}$$

The drift term $H(f_t, t)$ and control term $U(f_t)$ contribute to the covariance through their interaction with the derivative $f_t - \mathbb{E}[f_t]$. For a linearized approximation, we have

$$H(f_t, t) + U(f_t) \approx H(\mathbb{E}[f_t], t) + \frac{\partial H}{\partial f_t}(f_t - \mathbb{E}[f_t]) \tag{41}$$

Where $\frac{\partial H}{\partial f_t}$ is the Jacobian matrix of $H(f_t, t)$ evaluated as $\mathbb{E}[f_t]$. Substituting this into the drift contribution.

$$\mathbb{E}\Big[H(f_t, t) + U(f_t)(f_t - \mathbb{E}[f_t])^T\Big] \approx \mathbb{E}\Big[\frac{\partial H}{\partial f_t}(H(f_t, t) + U(f_t))(f_t - \mathbb{E}[f_t])^T\Big] \tag{42}$$

This simplifies to

$$\mathbb{E}\Big[H(f_t, t) + U(f_t)(f_t - \mathbb{E}[f_t])^T\Big] = \frac{\partial H}{\partial f_t}\sum_t \tag{43}$$

The diffusion term $G(f_t, t)$ contributes directly to the covariance as $\mathbb{E}[G(f_t, t)G(f_t, t)^T]$. The time evolution of the covariance matrix $\sum_t$ is theefore given by

$$\frac{d\sum_t}{dt} = \mathbb{E}\Big[G(f_t, t)G(f_t, t)^T\Big] + \mathbb{E}\Big[\frac{dH}{df_t}\Big]\sum_t + \sum_t \mathbb{E}\Big[\frac{dH}{df_t}\Big]^T \tag{44}$$

Which incorporates the diffusion term, contribution to variance, the drift's linearized effect on covariance and symmetry in the interaction of the drift with derivation from the mean. These derivations form the basis for deeper probabilistic analysis and enable us to quantify and manage the uncertainty inherent in complex dynamical systems.

5 Conclusion and Recommendations

5.1 Conclusion

This study has introduced a unified Bayesian framework for solving nonlinear stochastic differential equations (SDEs) with unknown dynamics, treating solution of the equations as a problem of probabilistic inference rather than deterministic computation. The proposed approach jointly reconstructs hidden state trajectories, estimates unknown drift and diffusion terms, and rigorously quantifies uncertainty. In doing so, it overcomes the fundamental limitations of classical solvers and filtering methods, which assume that system dynamics are known and typically neglect both modeling and discretization errors.

5.2 Recommendations

The findings recommend adopting Bayesian inference as the natural paradigm for analyzing nonlinear stochastic systems, particularly when governing equations or parameters are uncertain. By embedding uncertainty quantification directly into the solution process, the method avoids overconfident predictions and enables more reliable decision-making in applications ranging from control and engineering to finance and biology. Moreover, the framework replaces ad hoc parameter fitting with a coherent, data-driven identification of system components, ensuring consistent joint estimation of states and parameters. The resulting posterior distributions provide a principled basis for sensitivity analysis, reliability assessment, and risk-aware control strategies, making this approach a rigorous and practically valuable alternative to classical deterministic and approximate techniques.

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