Vol. 3(Iss. 1) 2026, pp.165-184, African Scientific Annual Review www.asarev.net ISSN: 3007-0902

Sensitivity and Optimal Control Analysis of Malaria Vaccination Model with Variable Controls

Alphablondy Achieng[1*]
Charles Wachira[2]
Benjamin Kikwai[3]

[1,2,3] Department of Mathematics and Statistics, Machakos University

[1*] alphablondyachieng20@gmail.com

https://doi.org/10.51867/asarev.3.1.16

Abstract

In this study a non-linear deterministic malaria vaccination model with optimal control is developed and analyzed. This improves a recent mathematical model for malaria with standard incidence by adding sensitivity analysis and optimal controls, where the goal is to obtain optimal control strategies through vaccination, treatment and personal protection with the aim of minimizing the number of infections in the population as well as treatment cost. The vaccine reproductive number R_v is computed using next generation matrix approach. Sensitivity analysis of R_v is conducted to identify the most influential parameters to be vaccine efficacy (ϵ), vaccination rate (α), mosquito biting rate (ϕ) and the natural death rate of the mosquitoes (μ_v). Maximum Principle of Pontryagin is applied to find the optimal control conditions of treatment efforts, vaccination efforts, and personal protection efforts on malaria transmission. The numerical simulations show that, combining control interventions strategies such as treatment efforts, vaccination efforts, and personal protection efforts such as (spraying of insecticides, use of mosquito treated bed nets, clearing of bushes and draining stagnant water around the homesteads etc) are necessary for the reduction of malaria. This study will help the policy makers in deducing control strategies such as increasing vaccination using vaccines of higher efficacy, increasing treatment and personal protection efforts for eradication of malaria infection.

MSC2010 Subject Classification: 92D30, 49K15, 93A30
Keywords: Sensitivity, Optimal Control Analysis, Reproduction number

1 Introduction

Malaria is a disease that is spread by an infected female Anopheles Mosquitoes to the human through biting. It is caused by parasites called *Plasmodium* that are carried by female anopheles Mosquitoes [9]. It is endemic in areas like Sub- Saharan Africa, the Caribbean and the pacific islands. There are five species of the *Plasmodium* (P) which infect humans as they are parasites which cause the

disease, they include, *P. falciparum, P. vivax, P. ovale, P. malariae* and *zoonotic P. knowelesi*. The most virulent is P. falciparum then followed by P. vivax which causes recurrent via dormant liver-stage parasites, that is responsible for hemolysis [12]. The clinical manifestations are fever, headache, nausea, fatigue and in severe cases, anemia, jaundice, cerebral malaria, multi-organ complications and death. Those at the greatest risk of contracting severe diseases and dying in case of infection are children, expectant mothers, and non-immune migrants [22].

Malaria vaccination works only for a specific species. The most infectious one is, *P. falciparum*, which is primarily treated using combine therapy with artemisinin ($ACTs$) [19], but infections of non-*falciparum* are susceptible to the action of chloroquine or other medications. The prevention strategies measures can be explained by the use of insecticides on the nets, indoor spraying, environmental management and in the recent past-vaccination. The first malaria vaccine is RTS,S/AS01 (Mosquirix) which has pilot projects in African countries like Kenya, Ghana and Malawi where the vaccine has partial protection, especially in children under five years old, in existence, conferring the vaccine with partial protection against malaria exposure [25]. Early studies suggest that combining vaccination with vector control and treatment interventions can significantly reduce malaria morbidity and mortality in endemic populations.

The research in [3] used awareness campaigns, insecticide, and treatment of infected people as time-dependent control. The basic reproduction number (R_0) obtained, the disease-free equilibrium (DFE) is globally asymptotically stable when ($R_0 < 1$). They applied Pontryagin's Minimum Principle to calculate the most efficient strategies within the context of available resources and included sensitivity analysis that revealed the most extensive parameters affecting dynamics of disease. Numerical results suggested awareness exercises, application of insecticides, and therapy represented the most economically successful technique in minimizing (R_0) .

Keno, Dano, and Makinde [14] explored the dynamics of malaria transmission in climatic variability by formulating a deterministic model that considers three time-dependent controls; treated nets ($ITNs$), antimalarial drug treatment and indoor residual spraying (IRS). Based on their analysis, they obtained the basic reproduction number (R_0) and demonstrated that disease-free equilibrium would be locally and globally asymptotically stable when ($R_0 < 1$). They developed an optimal control problem using Pontryagin's Maximum Principle and demonstrated through cost-effectiveness simulations that the synergy of treated bed nets and treatment is the most efficient and least expensive response to reduce the malaria burden under climatic variation conditions that change over time and space. Through cost-effectiveness simulations, they demonstrated that the combination of treated bed nets and treatment is the most efficient and the least expensive as the response to reducing the burden of malaria under conditions of climatic fluctuations that are changeable over time even though they did not include sensitivity analysis.

An optimal control model to evaluate the impact of combined approaches used in the intervention on malaria transmission between human and mosquito populations is developed and analyzed in [6] . The authors included time-dependent controls in their work to capture the use of insecticide-treated nets

($ITNs$), vaccination, and treatment, so that the systematic analysis of the effect of these interventions on the dynamics of the disease can be performed. They also examined the basic reproduction number R_0 and the stability of the disease-free equilibrium under optimal control. The research revealed that combining various controls has better results and generate meaningful changes in the level of infection at a lower cost.

Generally, population remains vulnerable to malaria at any given time notwithstanding the number of interventions put in place [27]. Therefore, there is need to study the malaria dynamics under vaccination and optimal control strategy while performing sensitivity to determine the most influential parameter in R_v so as to deduce the effective control strategies for malaria elimination. Based on a system of ordinary differential equations, a mathematical model is developed in this study to study the dynamics of malaria disease and deduce potential control strategies.

2 Model Formulation and Analysis

In this section, we revisit the malaria model studied in Achieng et al [1], and introduce optimal controls. The Malaria Free Equilibrium (MFE) point and reproduction number are shown.

2.1 Model Description and Formulation

Over the years, mathematical models have been used in the context of malaria to quantify the interactions between processes involving the host, parasite and the vector such as the rates of transmission, waning immunity and the dynamics of the vectors to generate patterns of transmission, and to estimate the effects of interventions such as the use of insecticide-treated nets, therapeutics and vaccination strategies [21]. In this study an optimal control non-linear differential-equations approach is used to develop a malaria $SVIRS - SI$ type model that takes into consideration vaccination, treatment and personal protection control strategies. The model assumes homogeneous mixing, ignores spatial heterogeneity. The total human population is taken to be N_h which is subdivided into four compartments namely, S_h the total susceptible humans, V_h vaccinated humans, I_h the number of infected humans, and R_h the number of recovered humans. The total number of individuals are recruited at a constant rate Λ_h which is taken to be susceptible human recruitment rate, the infection is acquired at a rate β_1 by the susceptible human from an infected mosquito, the vaccination rate of susceptible human is taken to be (α) whereas the vaccine efficacy is assumed to be (ϵ) and the human immunity waning rate is taken as (ω) as vaccine does not guarantee permanent immunity against malaria. Induced mosquito mortality effect is given as ψ=0.03. The disease recovery rate is given as (γ), where the vaccinated individual gets infected at a rate (β_2) where the transmission coefficient of the vaccinees $\beta_2 < \beta_1$ since the vaccinees are assumed to have acquired a vaccine induced immunity [1].

The total mosquito population is taken to be N_v which is divided into two compartments namely, S_v the total susceptible mosquitoes, I_v the total infected mosquitoes, the natural mortality rate is assumed to occur in the human and mosquito population at a rate of μ_h and μ_v respectively, the disease induced human death rate is given as δ as the susceptible mosquito recruitment rate is assumed

to be at a constant rate Λ_v. The mosquito biting rate is given as ϕ as the transmission rate from infected human to susceptible mosquito is assumed as κ. Forces of infection are $\phi\beta_1\frac{I_v}{N_v}$, $\phi\beta_2\frac{I_v}{N_v}$ and $\phi\kappa\frac{I_h}{N_v}$. The standard incidence rates are $\phi\beta_1\frac{S_hI_v}{N_v}$, $\phi\beta_2\frac{V_hI_v}{N_v}$ and $\phi\kappa\frac{S_vI_h}{N_h}$ where $\frac{I_h}{N_h}$ and $\frac{I_v}{N_v}$ are fractions of the infected human and mosquitoes respectively as in [11].

Table 1: **A descriptive summary of the model parameters, their unit values and sources.**

Symbol	**Description**	**Baseline Value**	**Sources**
Λ_h	Recruitment rate of humans	0.01547 day^{-1}	[7]
α	Vaccination rate of susceptible human	(0, 0.0085,1) varies	[23]
ϵ	Vaccine efficacy	(0, 0.5,1) varies	[5]
β_1	Transmission rate (mosquito $\rightarrow$ human)	0.048 day^{-1}	[18]
β_2	Transmission rate (mosquito $\rightarrow$ vaccinated)	0.004 day^{-1}	[8]
ϕ	Mosquito biting rate	0.33 day^{-1}	[18]
μ_h	Human natural death rate	4.74×10^{-5} day^{-1}	[18]
μ_v	mosquito natural death rate	0.1 day^{-1}	[18]
δ	Disease-induced human death rate	0.001 day^{-1}	[18]
γ	Recovery rate of infected humans	0.006 day^{-1}	[8]
ω	Loss of immunity rate	0.005 day^{-1}	[16]
Λ_v	Seasonal mosquito recruitment rate	0.0714 day^{-1}	[7]
κ	Transmission rate (Inf. human $\rightarrow$ mosquito)	0.48 day^{-1}	[18]

The above description from table (1) gives the following equations of non-linear ordinary differential equations:

$$\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h - \beta_1\phi\frac{S_hI_v}{N_v} - (\alpha+\mu_h)S_h + \omega R_h,\\
\frac{dV_h}{dt} &= \alpha S_h - (1-\epsilon)\beta_2\phi\frac{V_hI_v}{N_v} - \mu_h V_h,\\
\frac{dI_h}{dt} &= \beta_1\phi\frac{S_hI_v}{N_v} + (1-\epsilon)\beta_2\phi\frac{V_hI_v}{N_v} - (\mu_h+\gamma+\delta)I_h,\\
\frac{dR_h}{dt} &= \gamma I_h - \mu_h R_h - \omega R_h.\\
\frac{dS_v}{dt} &= \Lambda_v - \phi\kappa\frac{S_vI_h}{N_h} - \mu_v S_v,\\
\frac{dI_v}{dt} &= \phi\kappa\frac{S_vI_h}{N_h} - \mu_v I_v,
\end{aligned} \tag{1}$$

2.2 The Invariant Region

Since the model is a vector-host model, the solutions are positive and bounded as shown in Achieng et al [1]. The state variables $S_h(t)$, $V_h(t)$, $I_h(t)$, $R_h(t)$, $S_v(t)$ and $I_v(t)$ are always positive for every time $t \geq 0$, hence the population for humans and mosquitoes are bounded at $\frac{\Lambda_h}{\mu_h}$ and $\frac{\Lambda_v}{\mu_v}$ respectively.

2.3 Malaria Free Equilibrium (MFE) Point

The steady-state solution of the malaria population is the Malaria-Free Equilibrium (MFE) point. It is a state in which there is no malaria disease in the population, the malaria free equilibrium of model (1) is given by;

$$E^0 = \left(\frac{\Lambda_h}{\alpha + \mu_h},\ \frac{\alpha \Lambda_h}{\mu_h(\alpha + \mu_h)},\ 0,\ 0,\ \frac{\Lambda_v}{\mu_v},\ 0 \right). \tag{2}$$

2.4 The Reproductive Number

The basic reproduction number denoted by R_0, is defined as the average number of secondary infections produced by a single individual introduced into a fully susceptible population during an individual entire infectious period [4, 15]. The vaccine reproduction number denoted by R_v of model (1) is computed using the next generation matrix approach [17]. The vaccine reproduction number, R_v, is the threshold quantity that can predict the spread of the disease in a given population in the presence of vaccination. The basic reproduction number is given by the spectral radius of the matrix FV^{-1}, that is, $\mathcal{R}_0 = \rho(FV^{-1})$, where F and V are the next-generation matrices [17]. The operator FV^{-1}, known as the next-generation matrix, is constructed from the matrices of partial derivatives of F_i and V_i, where: F_i represents the rate of appearance of new infections in the i-th compartment, $V_i = V_i^- - V_i^+$ represents the rate of transfer (transition rate) into and out of the disease compartment i, with respect to the infected compartments (for example, I_h and I_v) evaluated at MFE. FV^{-1} is the reproduction number as computed in Achieng et al [1]. Therefore the vaccine reproduction number is given by;

$$R_v = \sqrt{\frac{\phi^2 \kappa \left(\beta_1 \mu_h + (1-\epsilon)\beta_2 \alpha\right)}{\mu_v(\alpha + \mu_h)\ (\mu_h + \gamma + \delta)}}. \tag{3}$$

$$R_v = R_0 \times \sqrt{\frac{\beta_1 \mu_h + (1-\epsilon)\beta_2 \alpha}{\beta_1(\alpha + \mu_h)}}. \tag{4}$$

If the vaccination rate of susceptible humans α=0, vaccine efficacy ϵ=0, then R_v=R_0, which is the basic reproduction number given by $R_0 = \sqrt{\frac{\phi^2 \kappa \beta_1}{\mu_v(\mu_h + \gamma + \delta)}}$.

The vaccine reproduction number, R_v, is the measure of the severity of an epidemic in the presence of vaccination and one of the most important parameters since it determines whether or not malaria will invade a population. Epidemiologically, if $R_v < 1$, then by definition, the infection does not spread in

the population. On the other hand, if $R_v > 1$, then the infection spreads in the population and may result into an epidemic.

3 Sensitivity Analysis

Sensitivity is the degree to which an input parameter to a model affect its output. Parameters which have a significant impact on the dynamics of infection are known as sensitive parameters [24]. Using the normalized forward sensitivity index of R_v, the sensitivity index with respect to the model parameter q is given by

$$\Gamma_q^{R_v} = \frac{\partial R_v}{\partial q} \times \frac{q}{R_v}. \tag{5}$$

Where q is the parameter whose sensitivity index is computed [24]. Table (2) gives a summary of the sensitivity indices of R_v evaluated at the baseline parameters values given in table (1).

From table (2) an increase of the mosquito biting rate (ϕ) by 1%, leads to an increase of the value of the vaccine reproduction number R_v by 1% hence the parameter ϕ is having a directly proportional relationship with R_v. An increase in the transmission rate from infected human to susceptible mosquitoes (κ) by 1% leads to an increase of the value of the R_v by 0.5%. 1% increase on vaccine efficacy (ϵ), decreases the R_v by 0.441%, when vaccination rate (α) is increased by 1% the R_v decreases by 0.0563%. Sensitivity analysis of model (1) parameters indicated that R_v is most sensitive to the mosquito biting rate (ϕ), the transmission rate from infected human to susceptible mosquitoes (κ), the vaccine efficacy (ϵ) and the vaccination rate (α). Sensitivity analysis results suggests that control measures are supposed to aim at decreasing the population of the vectors and the development of the vaccines of higher efficacy that would assist in enhancing human immune system.

Table 2: **Sensitivity indices of R_v with respect to model parameters**

Parameter	**Sensitivity Index $\Gamma_q^{R_v}$**
ϕ	+1.0000
κ	+0.5000
μ_v	−0.5000
β_2	+0.441
γ	−0.4258
ϵ	−0.441
δ	−0.0710
β_1	+0.059
α	−0.0563
μ_h	+0.0529

4 Optimal Control Analysis

In this section, an optimal control of various strategies of controlling malaria is conducted. The interest is to select the most appropriate scenario that when adequately put in place may assist in the containment of the malaria spread. Introducing into model(1) time dependent vaccination effort ($u_1(t)$), treatment effort ($u_2(t)$), and personal protection measures such as; (use of treated bed nets, application of mosquito repellents, spraying of insecticides, clearing of bushes and stagnant water around the homestead etc) ($u_3(t)$). Therefore model (1) becomes

$$\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h - \beta_1\phi(1-u_3)\frac{S_hI_v}{N_v} - (\alpha+u_1+\mu_h)S_h + \omega R_h,\\
\frac{dV_h}{dt} &= (\alpha+u_1)S_h - (1-\epsilon)\beta_2\phi(1-u_3)\frac{V_hI_v}{N_v} - \mu_hV_h,\\
\frac{dI_h}{dt} &= \beta_1\phi(1-u_3)\frac{S_hI_v}{N_v} + (1-\epsilon)\beta_2\phi(1-u_3)\frac{V_hI_v}{N_v} - (\mu_h+\gamma+u_2+\delta)I_h,\\
\frac{dR_h}{dt} &= (\gamma+u_2)I_h - (\mu_h+\omega)R_h,\\
\frac{dS_v}{dt} &= \Lambda_v - \phi\kappa(1-u_3)\frac{S_vI_h}{N_h} - (\mu_v+\psi u_3)S_v,\\
\frac{dI_v}{dt} &= \phi\kappa(1-u_3)\frac{S_vI_h}{N_h} - (\mu_v+\psi u_3)I_v,
\end{aligned} \tag{6}$$

The insecticides used for treating bed nets have insecticidal effects on mosquitoes thus reducing the number that attempts to feed on people in the sleeping areas with nets [26]. However, the mosquitoes can still feed on human outside this protective areas and so spraying of insecticides is included. Thus mosquito group is reduced at the rate $\mu_v + \psi u_3$ and $(1-u_3)$ reduces human mosquito interaction. The following objective function is created to accomplish this

$$J = W(u_1,u_2,u_3) = \int_0^T \left[A_1I_h + A_2I_v + \frac{1}{2}\left(W_1u_1^2 + W_2u_2^2 + W_3u_3^2\right)\right]dt, \tag{7}$$

where [0,T] is the entire time horizon over the control applied, the coefficients A_1, A_2, W_1, W_2, W_3, are positive weight functions to balance the factors, the goal is to minimize the number of infected humans and vectors A_1I_h, A_2I_v respectively, while minimizing the cost of controls $u_1(t)$, $u_2(t)$, $u_3(t)$ thus seeking an optimal control $u_1^{\max}$, $u_2^{\max}$, $u_3^{\max}$ such that $W(u_1^{\max},u_2^{\max},u_3^{\max}) = \min_{(u_1,u_2,u_3)\in U} W(u_1,u_2,u_3)$, such that (u_1,u_2,u_3) are measurable with $0 \le u_i \le 1$ for $i = 1,2,3$ [24], the term A_1I_h, A_2I_v are the costs of infection while $W_1u_1^2$, $W_2u_2^2$, $W_3u_3^2$ are the costs of use of vaccination effort, treatment efforts and personal protection efforts respectively, the necessary condition that an optimal control must satisfy come from the pontryagin's maximum principle [20], this principle converts (6) and (7) into a problem of minimizing pointwise a hamiltonian $\mathcal{H}$, with

respect to (u_1, u_2, u_3)

$$\begin{aligned}\mathcal{H} &= A_1 I_h + A_2 I_v + \frac{1}{2}(W_1 u_1^2 + W_2 u_2^2 + W_3 u_3^2) \\ &+ \lambda_1 \Big[\Lambda_h - \beta_1 \phi (1-u_3)\frac{S_h I_v}{N_v} - (\alpha + u_1 + \mu_h) S_h + \omega R_h\Big] \\ &+ \lambda_2 \Big[(\alpha + u_1) S_h - (1-\epsilon)\beta_2 \phi (1-u_3)\frac{V_h I_v}{N_v} - \mu_h V_h\Big] \\ &+ \lambda_3 \Big[\beta_1 \phi (1-u_3)\frac{S_h I_v}{N_v} + (1-\epsilon)\beta_2 \phi (1-u_3)\frac{V_h I_v}{N_v} - (\mu_h + \gamma + u_2 + \delta) I_h\Big] \\ &+ \lambda_4 \Big[(\gamma + u_2) I_h - (\mu_h + \omega) R_h\Big] \\ &+ \lambda_5 \Big[\Lambda_v - \phi \kappa (1-u_3)\frac{S_v I_h}{N_h} - (\mu_v + \psi u_3) S_v\Big] \\ &+ \lambda_6 \Big[\phi \kappa (1-u_3)\frac{S_v I_h}{N_h} - (\mu_v + \psi u_3) I_v\Big]. \end{aligned} \tag{8}$$

where the variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ are the adjoints to the state variables S_h, V_h, I_h, R_h, S_v, I_v, respectively. [10], gives the existence of optimal control due to the convexity of the integrand of J with respect to u_1, u_2, u_3, the boundedness of the state solutions and the lipschitz property of the state system with respect to the state variables. Applying Pontryagin's maximum principle [20] and the existence result for optimal control from [10], the following proposition is deduced

Proposition 1. *Given an optimal control* $u_1^{\max}$, $u_2^{\max}$, $u_3^{\max}$ *and solutions* S_h^*, V_h^*, I_h^*, R_h^*, S_v^*, I_v^* *of the corresponding state equation (6) that minimizes* $J = W(u_1, u_2, u_3)$*, there exist adjoint variables* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$*, satisfying,*

$$\begin{aligned}\lambda_1' &= \lambda_1 \left[\beta_1 \phi (1-u_3)\frac{I_v}{N_v} + \alpha + u_1 + \mu_h\right] - \lambda_2(\alpha + u_1) - \lambda_3 \beta_1 \phi (1-u_3)\frac{I_v}{N_v}, \\ \lambda_2' &= \lambda_2 \left[(1-\epsilon)\beta_2 \phi (1-u_3)\frac{I_v}{N_v} + \mu_h\right] - \lambda_3 (1-\epsilon)\beta_2 \phi (1-u_3)\frac{I_v}{N_v}, \\ \lambda_3' &= -A_1 + \lambda_3(\mu_h + \gamma + u_2 + \delta) - \lambda_4(\gamma + u_2) + (\lambda_5 - \lambda_6)\phi \kappa (1-u_3)\frac{S_v}{N_h}, \\ \lambda_4' &= \lambda_4(\mu_h + \omega) - \lambda_1 \omega, \\ \lambda_5' &= \lambda_5 \left[\phi \kappa (1-u_3)\frac{I_h}{N_h} + \mu_v + \psi u_3\right] - \lambda_6 \phi \kappa (1-u_3)\frac{I_h}{N_h}, \\ \lambda_6' &= -A_2 + (\lambda_1 - \lambda_3)\beta_1 \phi (1-u_3)\frac{S_h S_v}{N_v^2} + (\lambda_2 - \lambda_3)(1-\epsilon)\beta_2 \phi (1-u_3)\frac{V_h S_v}{N_v^2} + \lambda_6(\mu_v + \psi u_3) \end{aligned} \tag{9}$$

with transversality conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = \lambda_5(T) = \lambda_6(T) = 0. \tag{10}$$

thus, the optimal control takes the characterization form, which is given as follows:

$$
\begin{aligned}
u_1^{\max} &= \max\left\{0, \min\left\{\frac{S_h(\lambda_1-\lambda_2)}{W_1}, 1\right\}\right\},\\
u_2^{\max} &= \max\left\{0, \min\left\{\frac{I_h(\lambda_3-\lambda_4)}{W_2}, 1\right\}\right\},\\
u_3^{\max} &= \max\left\{0, \min\left\{\frac{1}{W_3}\Big[(\lambda_3-\lambda_1)\beta_1\phi\frac{S_hI_v}{N_v} + (\lambda_3-\lambda_2)(1-\epsilon)\beta_2\phi\frac{V_hI_v}{N_v}\right.\right.\\
&\quad \left.\left. + (\lambda_6-\lambda_5)\phi\kappa\frac{S_vI_h}{N_h} + \psi(\lambda_5S_v+\lambda_6I_v)\Big], 1\right\}\right\}.
\end{aligned}
\tag{11}
$$

Proof. The differential equations governing the adjoint variables are obtained by differentiating of Hamiltonian function (8), evaluating at the optimal control gives the adjoint system as in (9). On the interior of the control set where $\mathcal{H}$ can be minimized, hence;

$$
\begin{aligned}
0 = \frac{\partial H}{\partial u_1}\Big|_{u_1^{\max}} &= -W_1u_1 + \lambda_1S_h - \lambda_2S_h\\
\Rightarrow u_1^{\max} &= \frac{S_h(\lambda_1-\lambda_2)}{W_1}
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{\partial H}{\partial u_2}\Big|_{u_2^{\max}} &= -W_2u_2 + \lambda_3I_h - \lambda_4I_h\\
\Rightarrow u_2^{\max} &= \frac{(\lambda_3-\lambda_4)I_h}{W_2}
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{\partial H}{\partial u_3}\Big|_{u_3^{\max}} &= -W_3u_3 - \lambda_1\beta_1\phi\frac{S_hI_v}{N_v} - \lambda_2(1-\epsilon)\beta_2\phi\frac{V_hI_v}{N_v} + \lambda_3\beta_1\phi\frac{S_hI_v}{N_v}\\
&\quad + \lambda_3(1-\epsilon)\beta_2\phi\frac{V_hI_v}{N_v} - \lambda_5\phi\kappa\frac{S_vI_h}{N_h} + \psi\lambda_5S_v + \lambda_6\phi\kappa\frac{S_vI_h}{N_h} + \psi\lambda_6I_v
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow u_3^{\max} = \frac{1}{W_3}\Bigg[&(\lambda_3-\lambda_1)\beta_1\phi\frac{S_hI_v}{N_v} + (\lambda_3-\lambda_2)(1-\epsilon)\beta_2\phi\frac{V_hI_v}{N_v} + (\lambda_6-\lambda_5)\phi\kappa\frac{S_vI_h}{N_h}\\
&+\psi(\lambda_5S_v+\lambda_6I_v)\Bigg]
\end{aligned}
$$

Hence using the bounds yields equation system(11) as shown earlier and below

$$u_1^{\max} = \max\left\{0, \min\left\{\frac{S_h(\lambda_1-\lambda_2)}{W_1}, 1\right\}\right\},$$
$$u_2^{\max} = \max\left\{0, \min\left\{\frac{I_h(\lambda_3-\lambda_4)}{W_2}, 1\right\}\right\},$$
$$u_3^{\max} = \max\left\{0, \min\left\{\frac{1}{W_3}\Big[(\lambda_3-\lambda_1)\beta_1\phi\frac{S_hI_v}{N_v} + (\lambda_3-\lambda_2)(1-\epsilon)\beta_2\phi\frac{V_hI_v}{N_v} + (\lambda_6-\lambda_5)\phi\kappa\frac{S_vI_h}{N_h} + \psi(\lambda_5S_v+\lambda_6I_v)\Big], 1\right\}\right\}.$$

Now using the control argument $0 \le u_i \le 1$ for $i = 1, 2, 3$

$$u_1^{\max} = \begin{cases} 0, & \text{if } u_1^* \le 0, \\ u_1^*, & \text{if } 0 < u_1^* < 1, \\ 1, & \text{if } u_1^* \ge 1 \end{cases}$$
$$u_2^{\max} = \begin{cases} 0, & \text{if } u_2^* \le 0, \\ u_2^*, & \text{if } 0 < u_2^* < 1, \\ 1, & \text{if } u_2^* \ge 1 \end{cases} \tag{12}$$
$$u_3^{\max} = \begin{cases} 0, & \text{if } u_3^* \le 0, \\ u_3^*, & \text{if } 0 < u_3^* < 1, \\ 1, & \text{if } u_3^* \ge 1 \end{cases}$$

Which leads to the following hence bounding them yields the characteristics equations system(11)

$$u_1^* = \frac{S_h(\lambda_1-\lambda_2)}{W_1}$$
$$u_2^* = \frac{(\lambda_3-\lambda_4)I_h}{W_2}$$

$$u_3^* = \frac{1}{W_3}\Big[(\lambda_3-\lambda_1)\beta_1\phi\frac{S_hI_v}{N_v} + (\lambda_3-\lambda_2)(1-\epsilon)\beta_2\phi\frac{V_hI_v}{N_v} + (\lambda_6-\lambda_5)\phi\kappa\frac{S_vI_h}{N_h} + \psi\lambda_5S_v + \psi\lambda_6I_v\Big]$$

□

Due to the priori boundedness of the state and adjoint functions and the resulting lipschitz structure of (6), the uniqueness of the optimal control for small time (T) can be obtained, following the technique

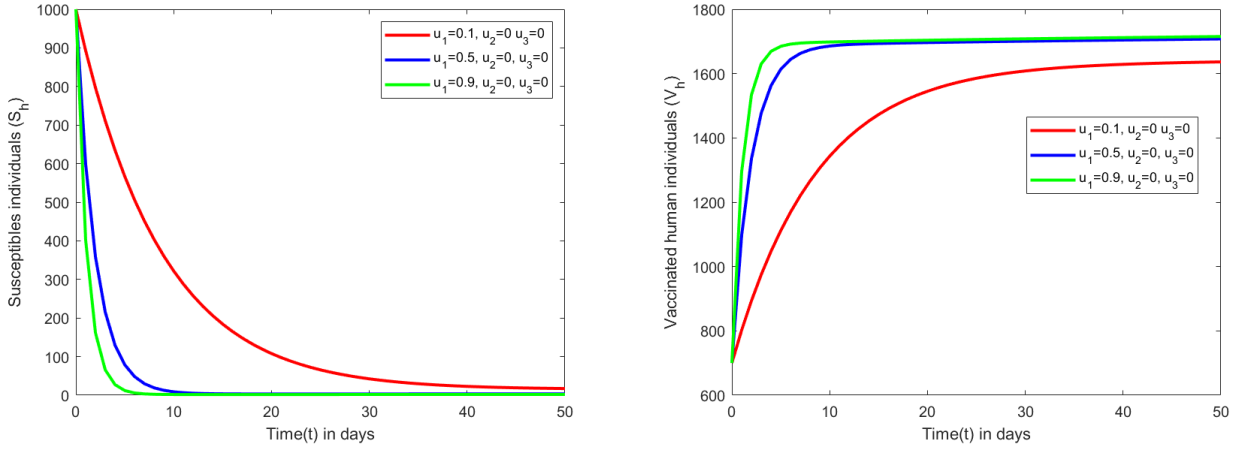
from [2]. The uniqueness of the optimal control follows from the uniqueness of the optimality system, which consists of (6) and (9), (10), with characterization(11). There is a restriction on the length of the time interval in order to guarantee the uniqueness of the optimality system as in [13]. This smallness restriction is due to the opposite time orientations of the optimality system. The state problem has initial values and the adjoint problem has final value.

5 Numerical Simulations and Discussion

Simulation analysis of the model (1) are presented to graphically illustrate the behaviour of the solutions of the model with varying parameter values. This helps in deducing control strategies against malaria disease. The variation of these optimal control strategies u_1, u_2 and u_3 to determine the numerical simulation of the controls, which are vaccination, treatment and personal protection respectively. Malaria disease control strategies are being used including the use of insecticide treated mosquito nets ($ITNs$), clearing of bushes around homesteads, insecticide spraying, vaccination and treatment [8, 21]. For the purpose of the simulation, the initial population are taken to be $S_h(0)$=1000, $V_h(0)$=700, $I_h(0)$=500, $R_h(0)$=300, $S_v(0)$=1500, $I_v(0)$=1000, with a high mosquito population compared to human population, this is to allow the spread of infection [7].

Scenario 1 varying $u_1 \neq 0$, $u_2 = 0$, $u_3 = 0$

The effect of varying vaccination effort on susceptible and vaccinated humans when there is no treatment and personal protection efforts.



(a) Vaccination effect on S_h

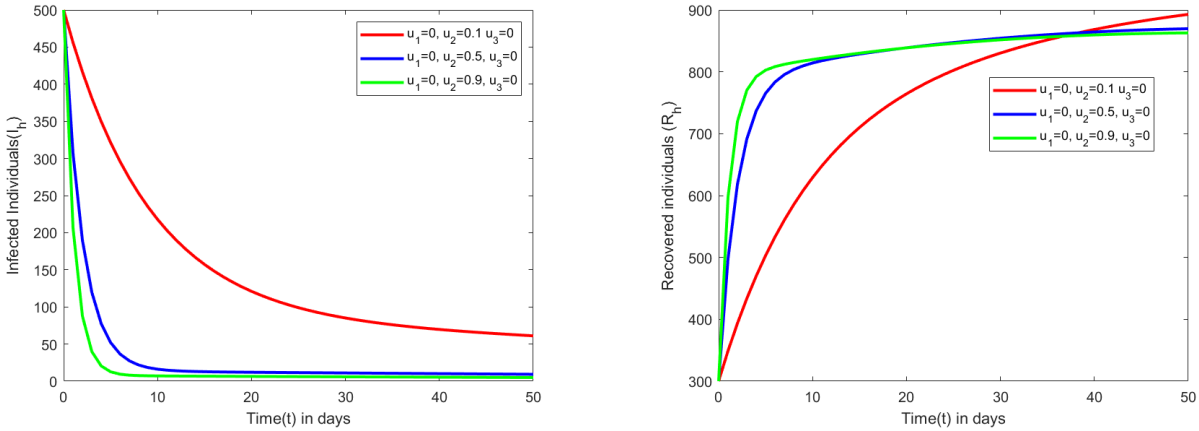
(b) Vaccination effect on V_h

Figure 1: The effect of variation in vaccination (u_1) effort when $u_2 = u_3 = 0$ on susceptible human (S_h) and vaccinated human (V_h).

Figure (1) shows the effect of vaccination on susceptible human (S_h) and vaccinated human (V_h), when the vaccination rate is high, there is a sharp decrease on the susceptible human population as shown in figure (1a) as they move to vaccinated class as shown in figure (1b) hence a sharp increase on the vaccinated population as seen, this later affects the infectious mosquito population (I_v) as there is reduced number of (I_h). This trend suggests that there would be reduced number of infectious mosquitos, reducing the spread of malaria in the population.

Scenario 2 varying $u_2 \neq 0$, $u_1 = 0$, $u_3 = 0$

The effect of varying treatment effort on infected and recovered humans when there is no vaccination effort and personal protection.



(a) Treatment effect on I_h

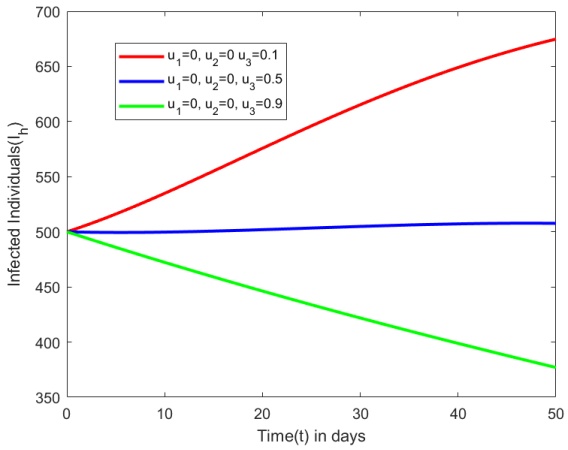
(b) Treatment effect on R_h

Figure 2: Simulation of the effect of varying treatment effort (u_2) when ($u_1 = u_3 = 0$) on infected human (I_h) and recovered human (R_h).

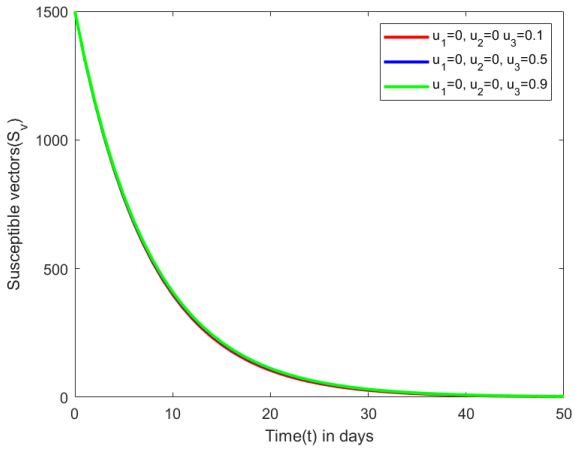
Figure (2) shows the effect of varying treatment effort (u_2) on infected human (I_h) and recovered human (R_h) when ($u_1 = u_2 = 0$), when treatment rate (u_2) is around 90%, there is a sharp decrease on the infected population (I_h) as shown in figure (2a) and a sharp increase on the recovered human (R_h) as shown in figure (2b) since the infected individuals move to recovered humans after treatment. This affect the susceptible mosquitoes (S_v) since the point of infection is curtailed reading to reduction in malaria transmission. This trend occurs when more infected individuals are treated, lowering the likelihood of malaria transmission.

Scenario 3 varying $u_3 \neq 0$, $u_1 = 0$, $u_2 = 0$

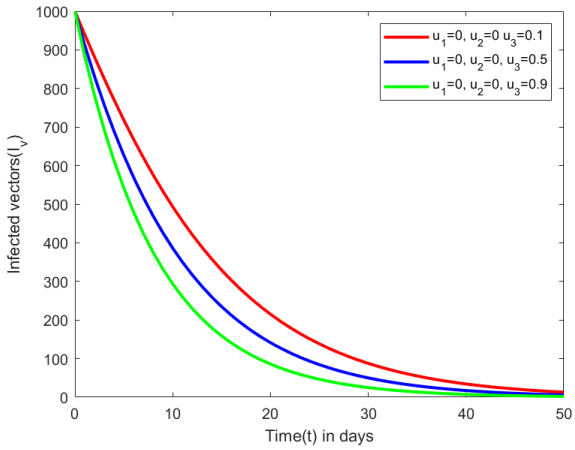
The effect of varying personal protection effort on infected humans, susceptible and infected vectors.



(a) Personal protection effect on I_h



(b) Personal protection effect on S_v



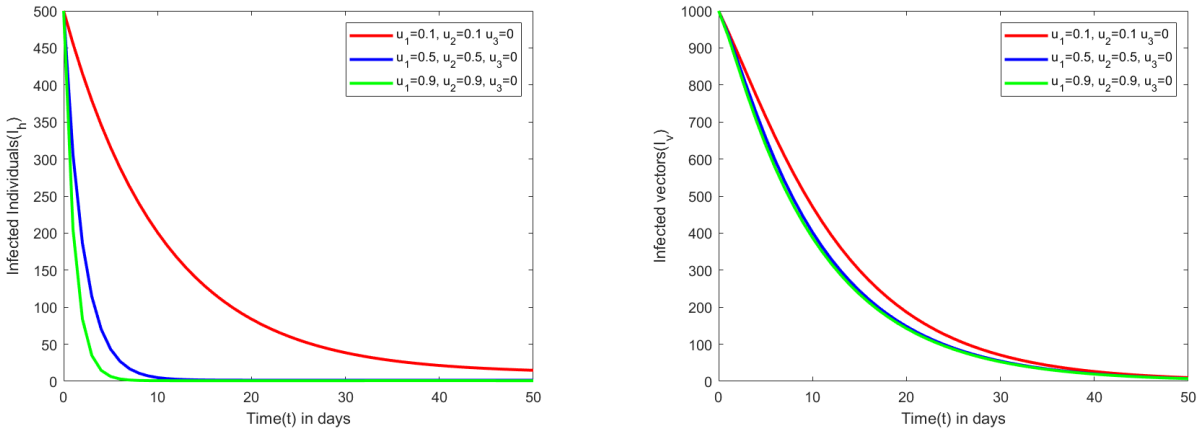
(c) Personal protection effect on I_v

Figure 3: The effect of varying personal protection effort (u_3) when ($u_1 = u_2 = 0$) on infectious human (I_h), susceptible mosquitoes (S_v) and infectious mosquito (I_v).

Figure (3) shows the effect of personal protection (u_3) effort, on infected human (I_h), susceptible mosquitoes (S_v), infected mosquitoes (I_v), when $u_1 = u_2 = 0$. When the personal protection (u_3) effort is high, then the infected individuals as shown in figure (3a) decreases at a higher rate compared to when the personal protection rate is low, when there is higher (u_3), then (S_v) as shown in figure (3b) and (I_v) as shown in figure (3c) decreases at a sharp rate. This is attributed by the use of insecticides, clearing of bushes and draining of stagnant water around homesteads and use of bed nets, which significantly induces mosquito mortality rate. Consequently, reducing the human-mosquito interactions leading to a lower risk of infection.

Scenario 4 varying $u_1 \neq 0$, $u_2 \neq 0$, $u_3 = 0$

The effect of varying vaccination and treatment efforts on infected humans and infected vectors.



(a) Vaccination and treatment effect on I_h

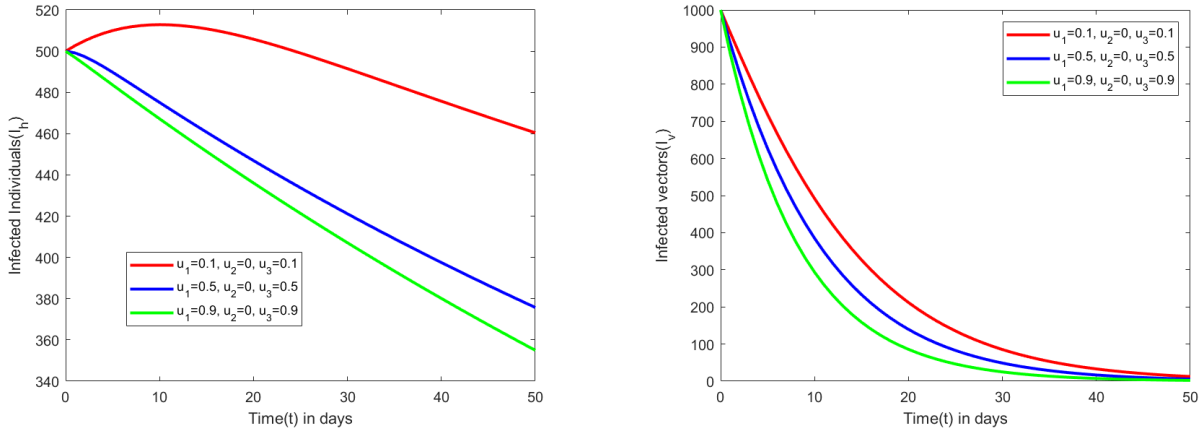
(b) Vaccination and treatment effect on I_v

Figure 4: The effect of varying vaccination (u_1) and treatment (u_2) efforts when personal protection ($u_3 = 0$) on infectious human (I_h) and infected mosquitoes (I_v).

Figure (4) shows the effects of varying vaccination (u_1) and treatment (u_2) efforts, on the infected human (I_h) and infected mosquitoes (I_v) when ($u_3 = 0$), when vaccination (u_1) and treatment (u_2) efforts are increased to around 90%, then there is a sharp decrease on the infected human (I_h) as shown in figure (4a), on the other hand there is a gradual decrease on the infected vectors (I_v) as shown in figure (4b) as the interaction between susceptible vector and infected human decreases. This trend occurs because a large portion of human population become protected through vaccination and treatment, hence the likelihood of transmission decreases leading to a reduction in malaria transmission cases.

Scenario 5 varying $u_1 \neq 0$, $u_3 \neq 0$, $u_2 = 0$

The effect of varying vaccination efforts and personal protection efforts on infected human and infected vectors.



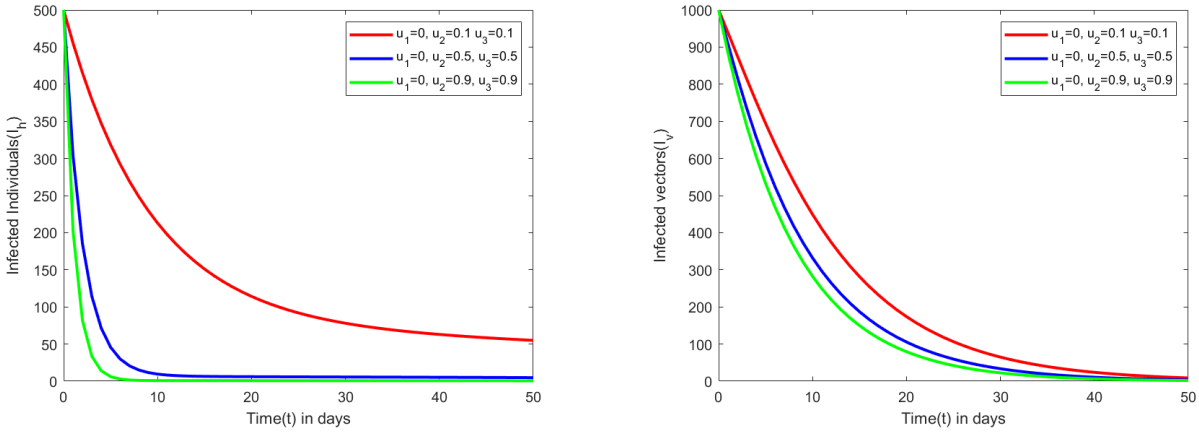
(a) Vaccination and personal protection effect on I_h (b) Vaccination and personal protection effect on I_v

Figure 5: The effect of varying vaccination (u_1) and personal protection (u_3) efforts when treatment ($u_2 = 0$) on infectious human (I_h) and infected mosquitoes (I_v).

Figure (5) shows the effects of varying control strategies vaccination (u_1) and personal protection (u_3) efforts, on the infected human (I_h) and infected mosquitoes (I_v) when ($u_2 = 0$). When vaccination (u_1) and personal protection (u_3) efforts are increased to around 90%, then there is a sharp decrease on the infected human (I_h) as shown in figure (5a), on the other hand there is a gradual decrease on the infected vectors (I_v) as shown in figure (5b) as the interaction between susceptible vector and infected human decreases. Vaccination helps to boost immunity, reducing the probability of getting infected, while protection measures limits the interaction of human with infected vector, leading to reduction in malaria transmission.

Scenario 6 varying $u_2 \neq 0$, $u_3 \neq 0$, $u_1 = 0$

The effect of varying treatment and personal protection efforts on infected human and vectors.



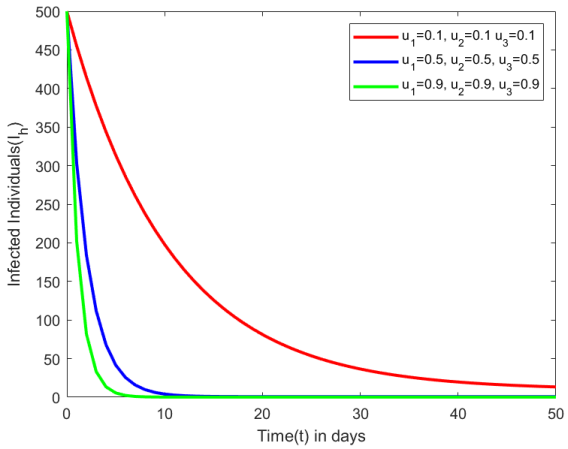
(a) Treatment and personal protection effect on I_h (b) Treatment and personal protection effect on I_v

Figure 6: Simulation of effect of varying treatment (u_2) and personal protection (u_3) efforts when vaccination effort ($u_1 = 0$) on infectious human (I_h) and infected mosquitoes (I_v).

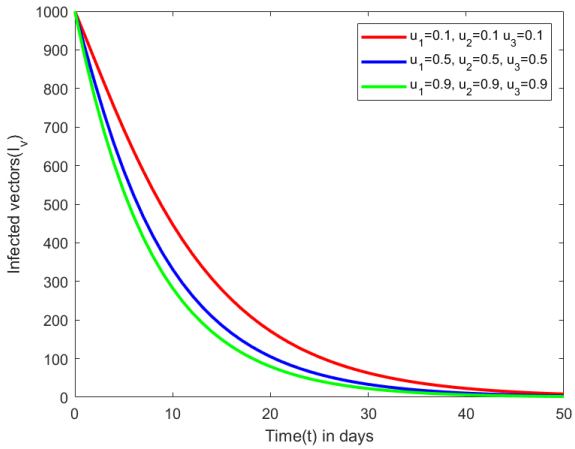
Figure (6) shows the effects of treatment (u_2) and personal protection (u_3) efforts, on the infected human (I_h) and infected mosquitoes (I_v) when ($u_1 = 0$), when treatment (u_2) and personal protection (u_3) efforts are increased to around 90%, then there is a sharp decrease on the infected human (I_h) as shown in figure (6a), on the other hand there is a gradual decrease on the infected vectors (I_v) as shown in figure (6b) as the interaction between susceptible vector and infected human decreases. Increased efforts on treatment reduces the number of infected individuals and the impact of personal protection reduces the vector population. As a result, the likelihood of malaria transmission decreases in human population.

Scenario 7 varying $u_1 \neq 0$, $u_2 \neq 0$, $u_3 \neq 0$

The effect of varying vaccination, treatment and personal protection efforts on infected human and infected vectors.



(a) Combined efforts of Vaccination (u_1), treatment (u_2) and personal protection (u_3) effect on I_h



(b) Combined effort Vaccination (u_1), treatment (u_2) and personal protection (u_3) effect on (I_v)

Figure 7: Simulation of effect of varying vaccination (u_1), treatment (u_2) and personal protection (u_3) efforts on infectious human (I_h) and infected mosquitoes (I_v).

Figure (7) shows the effects of varying vaccination (u_1), treatment (u_2) and personal protection (u_3) efforts, on the infected human (I_h) and infected mosquitoes (I_v), when vaccination (u_1), treatment (u_2) and personal protection (u_3) efforts are increased to around 90%, then there is a sharp decrease on the infected human (I_h) as shown in figure (7a), on the other hand there is a gradual decrease on the infected vectors (I_v) as shown in figure (7b) as the interaction between susceptible vector and infected human decreases. This suggests that combined use of these efforts, would ultimately lead to a reduction in malaria spread.

6 Conclusion

An optimal control malaria model with vaccination is presented and analysed. The vaccine reproduction number R_v is computed based on the next generation matrix approach. Sensitivity analysis is carried out on the parameters of R_v to ascertain parameters that are most influential for the disease to invade population. The results suggests that the mosquito biting rate ϕ, vaccine efficacy (ϵ), vaccination rate (α), and the natural death rate of the mosquitoes (μ_v) are the most influential parameter of R_v, therefore a need for the control measures to target reducing the biting rate of the mosquitoes. The optimal control efforts of treatment, vaccination and personal protection against malaria is computed using Pontriagin's Maximum Principle. From numerical simulations as shown in Figures 1-7, control strategies such as use of insecticide-treated mosquito nets, clearing of bushes near homesteads, spraying of insecticides, vaccination and treatment are deduced. The findings underscores the necessity of emphasizing on personal protection measures alongside treatment and vaccination efforts, to accomplish a thorough malaria control. A number of proposed control strategies were

identified by studies such as [8, 21]. Future studies may incorporate spatial factors and perform cost-effectiveness analysis of the control strategies.

References

[1] Achieng, A., Wachira, C., & Kikwai, B. (2026). Mathematical analysis of an extended SIRS-SI malaria model incorporating seasonal vector recruitment with standard incidence. *African Journal of Empirical Research*, 7(2), 1400-1419. https://doi.org/10.51867/ajernet.maths.7.2.122

[2] Ahmed, J. M., Tilahun, G. T., & Degefa, S. T. (2024). In-host fractional order model for malaria parasite dynamics with immune system. *Modeling Earth Systems and Environment*, 10(3), 4185-4205. https://doi.org/10.1007/s40808-024-02004-4

[3] Al Basir, F., & Abraha, T. (2023). Mathematical modelling and optimal control of malaria using awareness-based interventions. *Mathematics*, 11(7), 1687. https://doi.org/10.3390/math11071687

[4] Anderson, R. M., & May, R. M. (1991). *Infectious diseases of humans: dynamics and control*. Oxford university press.

[5] Akowe, E., Ahman, Q. O., Agbata, B. C., Joseph, S. O., Senewo, E. O., Danjuma, A. Y., & Yahaya, D. J. (2025). A novel malaria mathematical model: integrating vector and non-vector transmission pathways. *BMC Infectious Diseases*, 25(1), 322. https://doi.org/10.1186/s12879-025-10653-8

[6] Appiah, P., Amankwah, H., & Benyah, F. (2024). Optimal Control of a Malaria Transmission Model with Saturated Incidence. *Journal of Mathematics and Statistics* 2024, Volume (20): 18.36. DOI: 10.3844/jmssp.2024.18.36

[7] Charles, W. M., George, L. O., & Malinzi, J. (2018). A spatiotemporal model on the transmission dynamics of Zika virus disease.*Asian Research Journal of Mathematics*, 10(4), 1-15. DOI:10.9734/ARJOM/2018/43944

[8] Duve, P., Charles, S., Munyakazi, J., Lühken, R., & Witbooi, P. (2024). A mathematical model for malaria disease dynamics with vaccination and infected immigrants. *Math. Biosci. Eng.*, 21(1), 1082-1109. DOI: 10.3934/mbe.2024045

[9] Fikadu, M., & Ashenafi, E. (2023). Malaria: An Overview. *Infection and Drug Resistance*, 16, 3339–3347. https://doi.org/10.2147/IDR.S405668

[10] Fleming, W. H., & Rishel, R. W. (2012). *Deterministic and stochastic optimal control*. Springer Science & Business Media.

[11] Guo, S. B., He, M., & Cui, J. A. (2023). Global stability of a time-delayed malaria model with standard incidence rate. *Acta Mathematicae Applicatae Sinica, English Series, 39(2), 211-221.* https://doi.org/10.1007/s10255-023-1042-y

[12] Habtamu, K., Petros, B., & Yan, G. (2022). Plasmodium vivax: the potential obstacles it presents to malaria elimination and eradication. *Tropical Diseases, Travel Medicine and Vaccines*, 8(1), 27. https://doi.org/10.1186/s40794-022-00185-3

[13] Joshi, H. R. (2002). Optimal control of an HIV immunology model. *Optimal control applications and methods*, 23(4), 199-213. DOI: 10.1002/oca.710

[14] Keno, T. D., Dano, L. B., & Makinde, O. D. (2022). Modeling and optimal control analysis for malaria transmission with role of climate variability. *Computational and Mathematical Methods*, 2022(1), 9667396. https://doi.org/10.1155/2022/9667396

[15] Lotfi, E. M., Maziane, M., Hattaf, K., & Yousfi, N. (2014). Partial differential equations of an epidemic model with spatial diffusion. *International Journal of Partial Differential Equations*, 2014(1), 186437. http://dx.doi.org/10.1155/2014/186437

[16] Ochieng, F. O. (2024). SEIRS model for malaria transmission dynamics incorporating seasonality and awareness campaign. *Infectious Disease Modelling*, 9(1), 84-102. https://doi.org/10.1016/j.idm.2023.11.010

[17] Ochwach, J. O., Mark, O. O., & Murwayi, A. L. M. (2023). On Basic Reproduction Number R_0: Derivation and Application. *Journal of Engineering and Applied Sciences Technology. SRC/JEAST-234*. DOI: doi. org/10.47363/JEAST/2023 (5), 173, 2-14.

[18] Osman, M. A., Ebenezer, A., Hassan, N. A., & Yang, C. (2019). Sensitivity analysis of mathematical model for malaria transmission with saturated incidence rate. *Journal of Scientific Research and Reports*, 22(3), 1-10. DOI: 10.9734/JSRR/2019/46131

[19] Okombo, J., & Fidock, D. A. (2025). Towards next-generation treatment options to combat Plasmodium falciparum malaria. *Nature Reviews Microbiology*, 23(3), 178-191. doi:10.1038/s41579-024-01099-x.

[20] Pontryagin, L. S. (2018). *Mathematical theory of optimal processes*. Routledge.

[21] Qu, Z., Patterson, D., Zhao, L., Ponce, J., Edholm, C. J., Prosper Feldman, O. F., & Childs, L. M. (2025). Mathematical modeling of malaria vaccination with seasonality and immune feedback. *PLOS Computational Biology*, 21(5). https://doi.org/10.1371/journal.pcbi.1012988

[22] Siagian, F. E. (2020). Malaria epidemiology: specific vulnerable group in the population. *International Journal of Research and Reports in Hematology*, 3(2), 41-52. https://www.journalijr2h.com/index.php/IJR2H/index

[23] Tchoumi, S. Y., Chukwu, C. W., Diagne, M. L., Rwezaura, H., Juga, M. L., & Tchuenche, J. M. (2022). Optimal control of a two-group malaria transmission model with vaccination. *Network Modeling Analysis in Health Informatics and Bioinformatics*, 12(1), 7. doi: 10.1007/s13721-022 00403-0

[24] Wachira, C. M., Lawi, G. O., & Omondi, L. O. (2022). Sensitivity and Optimal Control Analysis of an Extended SEIR COVID-19 Mathematical Model. *Journal of Mathematics*, 2022(1), 1476607. https://doi.org/10.1155/2022/1476607

[25] Walldorf, J. A., Mayers, G. F., Amponsa-Achiano, K., Jalang'o, R., Chisema, M., Khalayi, L., & Okine, R. N. (2025). Post Introduction Evaluation (PIE) of the malaria vaccine introduced in three pilot countries (Ghana, Kenya, and Malawi) in 2021. *Malaria Journal*, 24(1), 333. https://doi.org/10.1186/s12936-025-05590-5

[26] World Health Organization. (2025). *Malaria surveillance, monitoring and evaluation: A reference manual*. World Health Organization.

[27] World Health Organization. (2024). *Global malaria programme operational strategy*, World Health Organization.